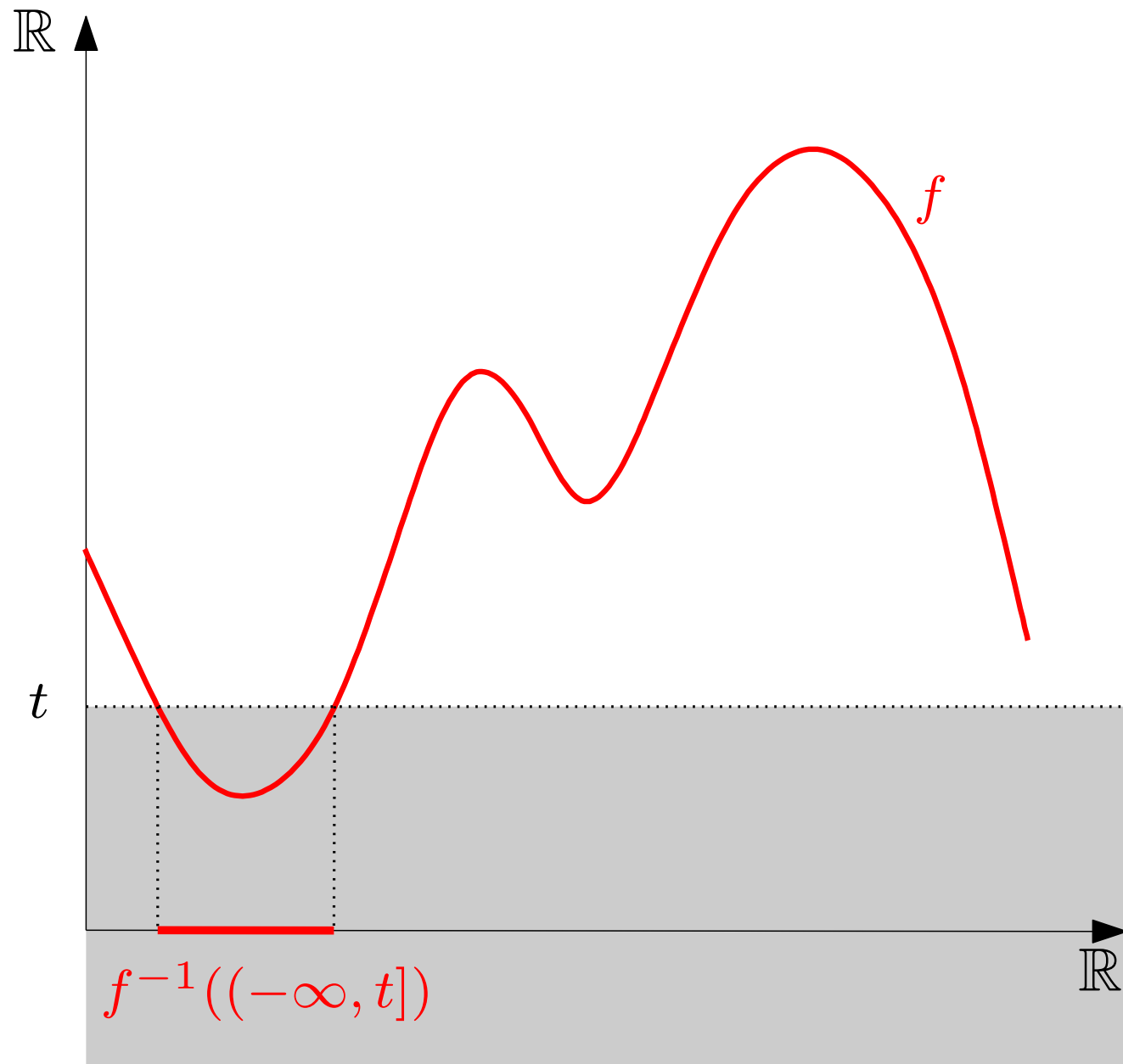# Persistent Homology

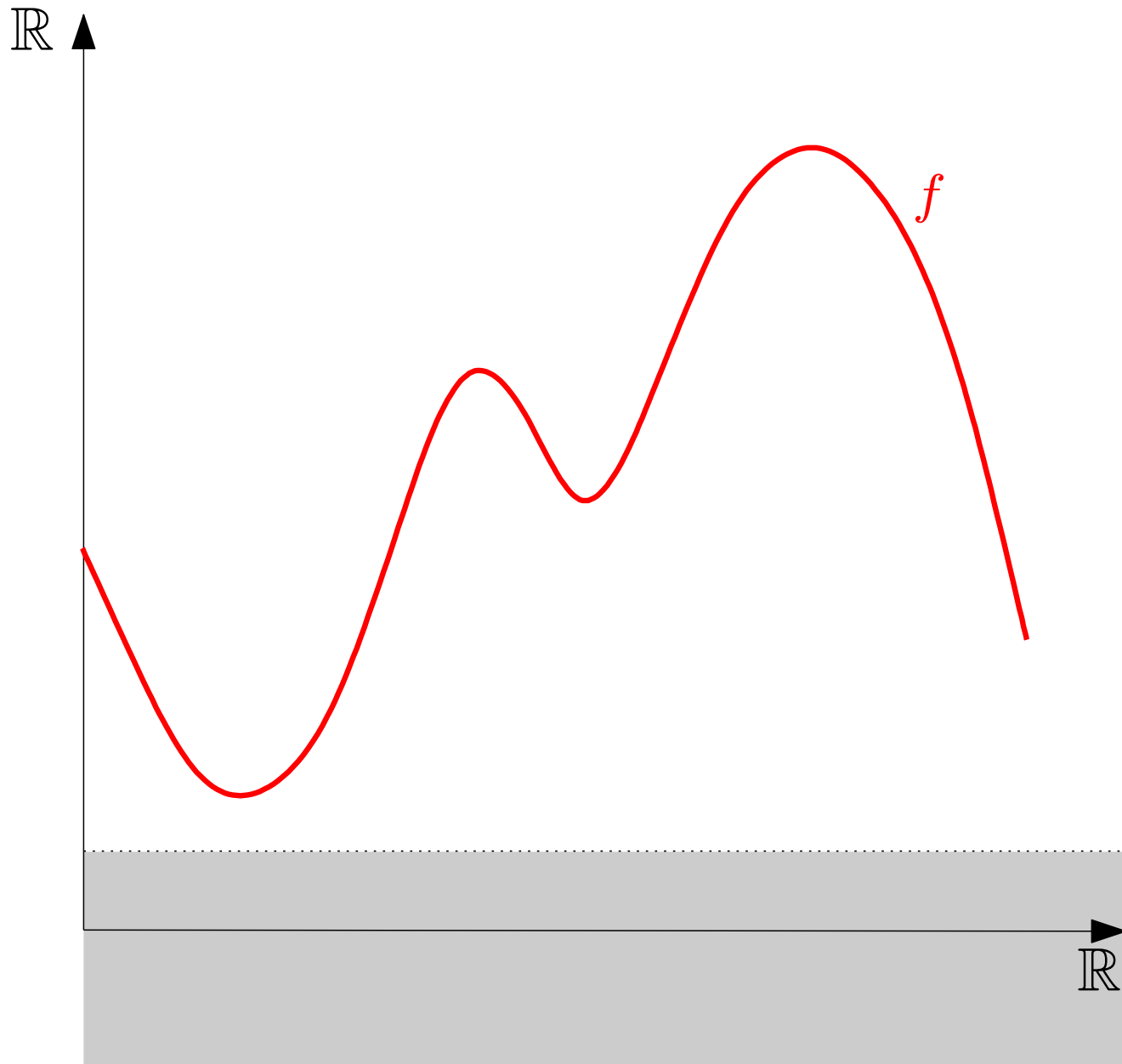# Smooth Real Function

Inside the black box:

- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family
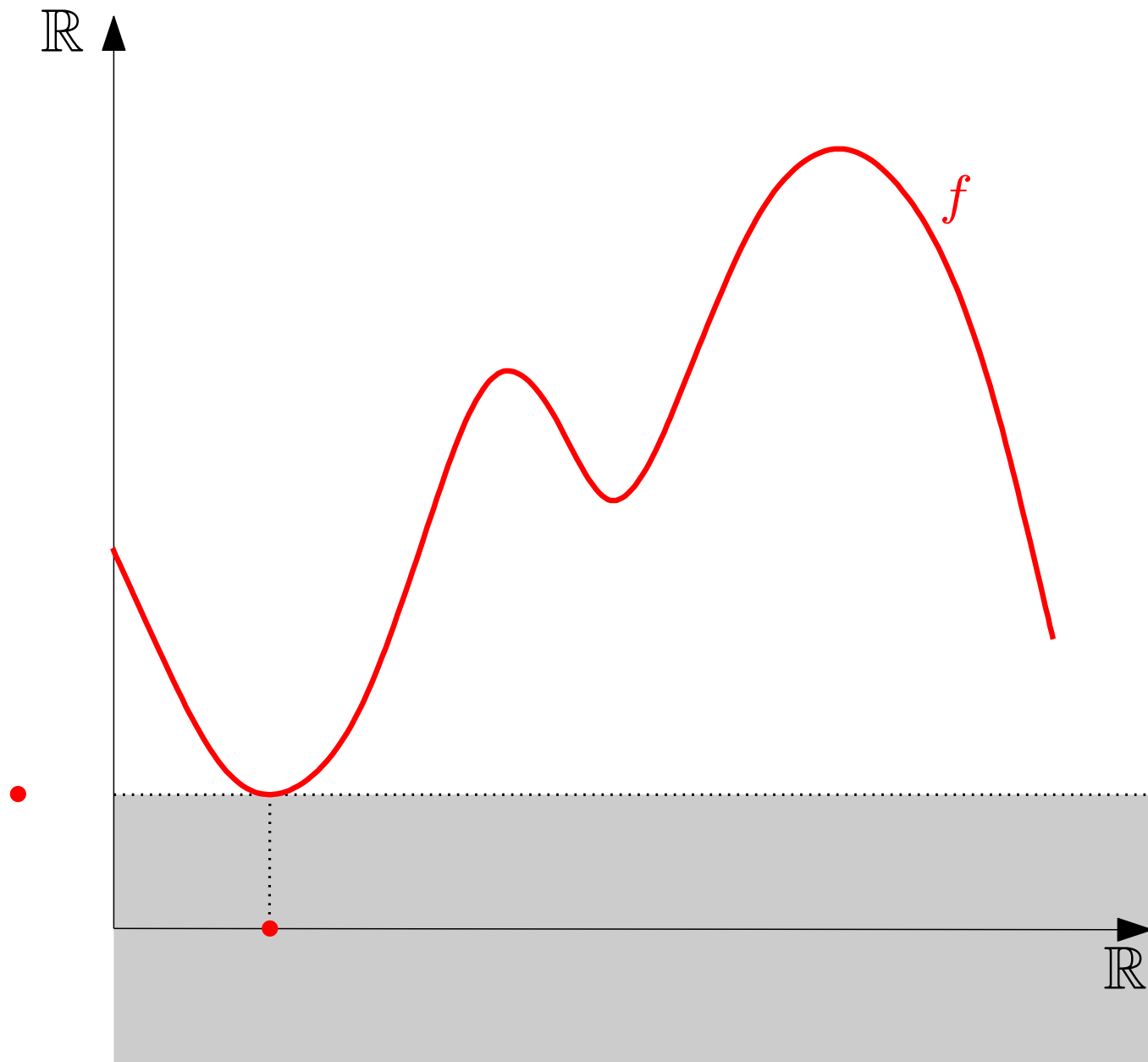

2

Smooth Real Function

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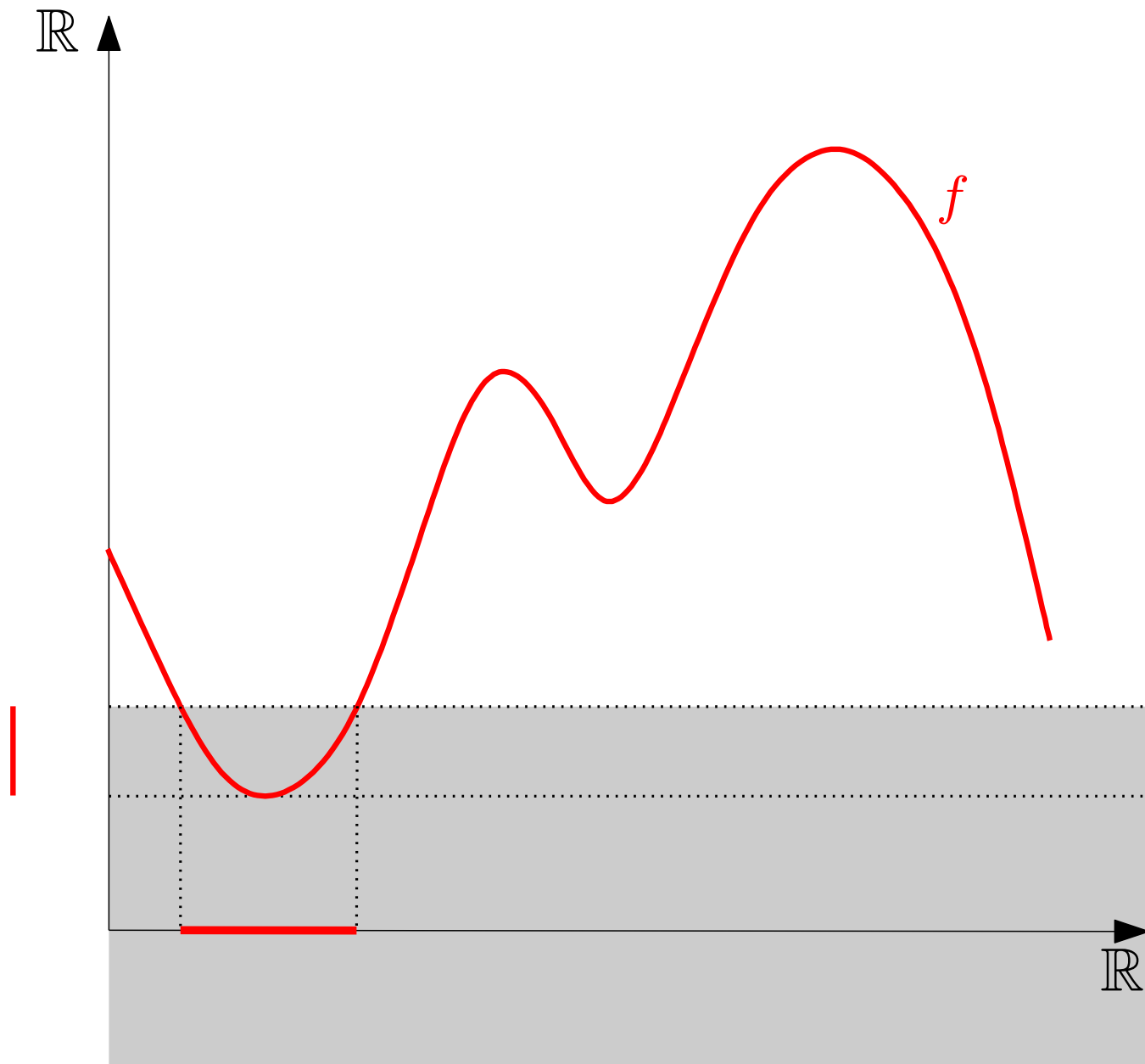# Smooth Real Function

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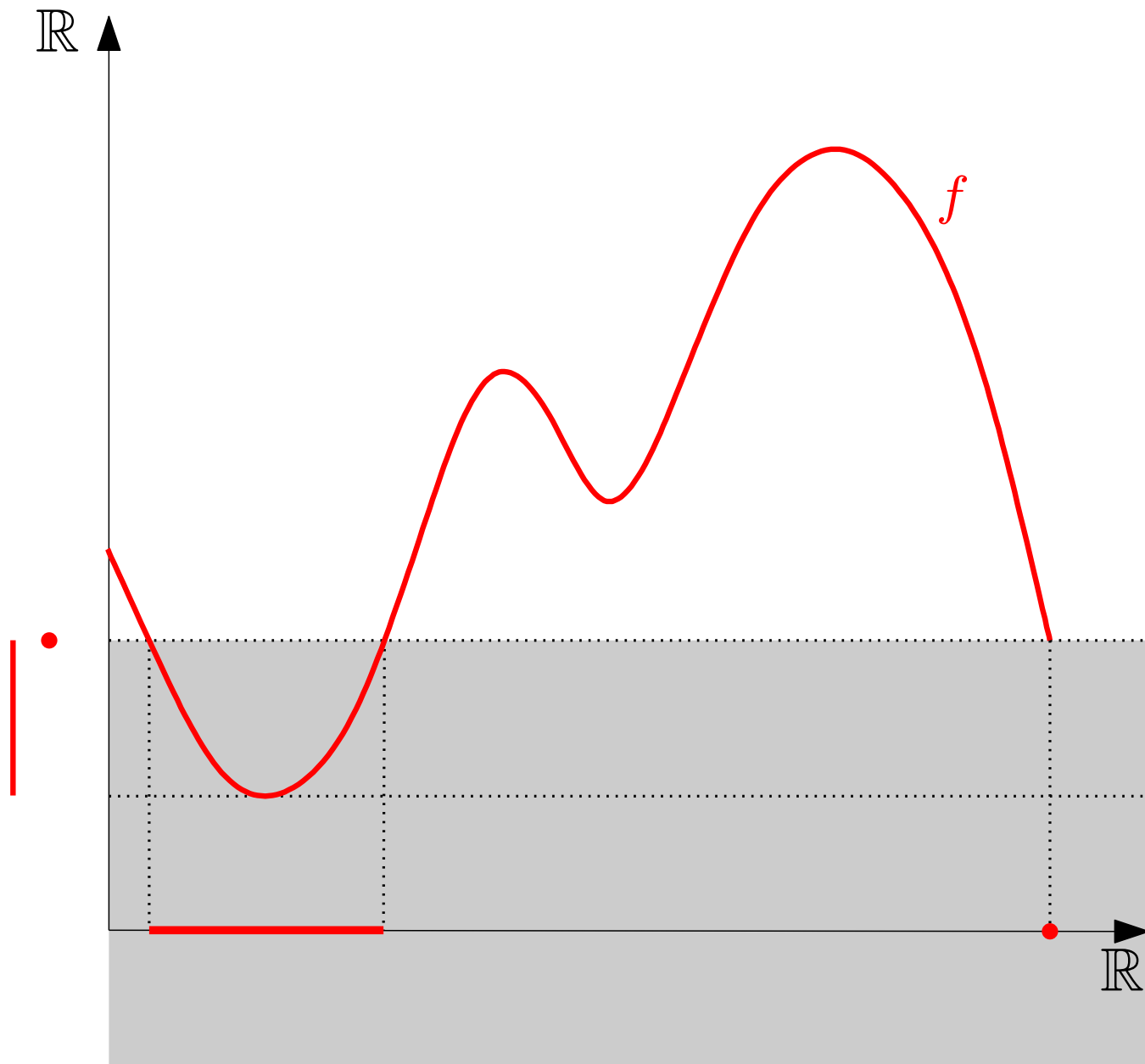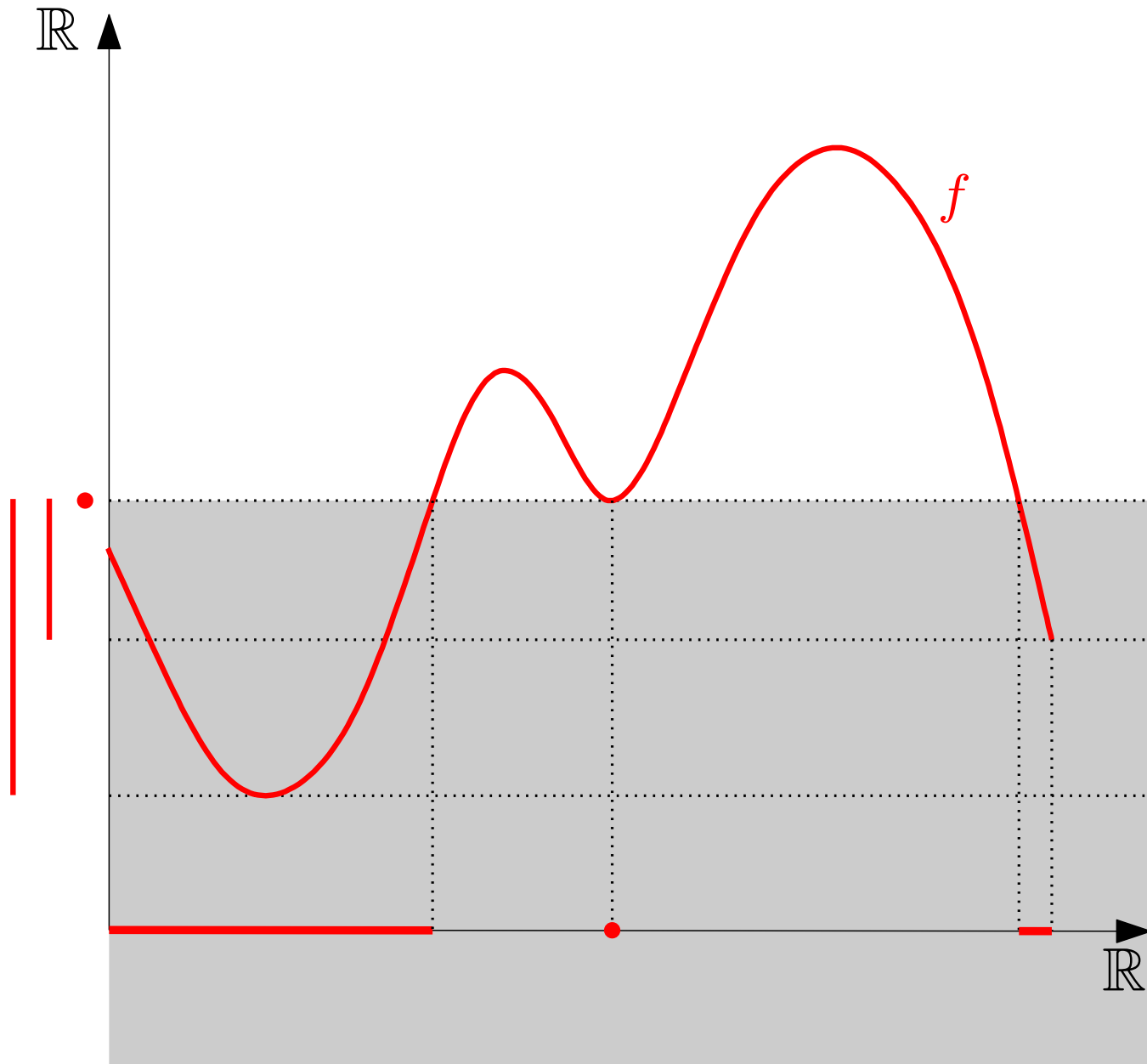# Smooth Real Function

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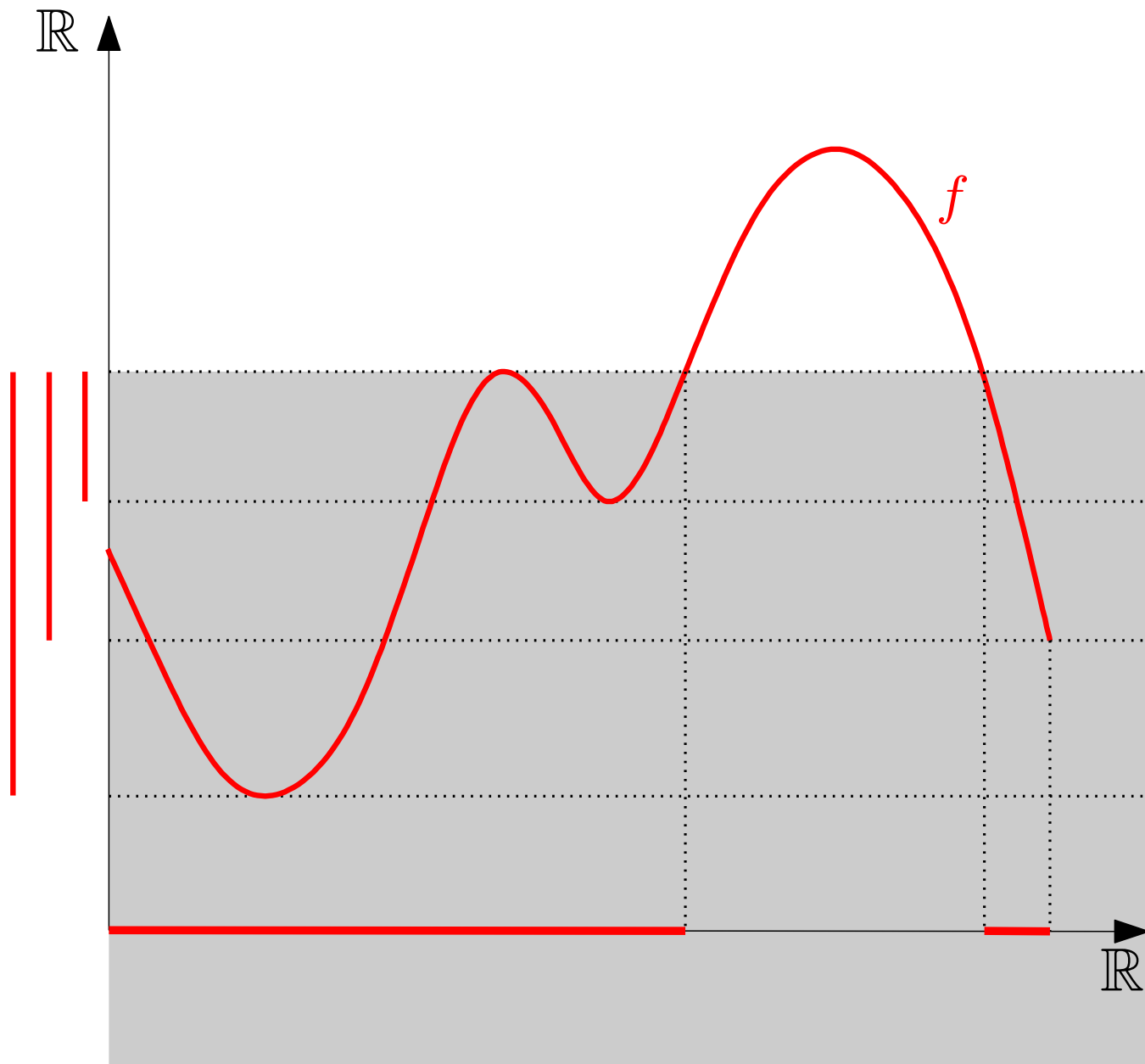
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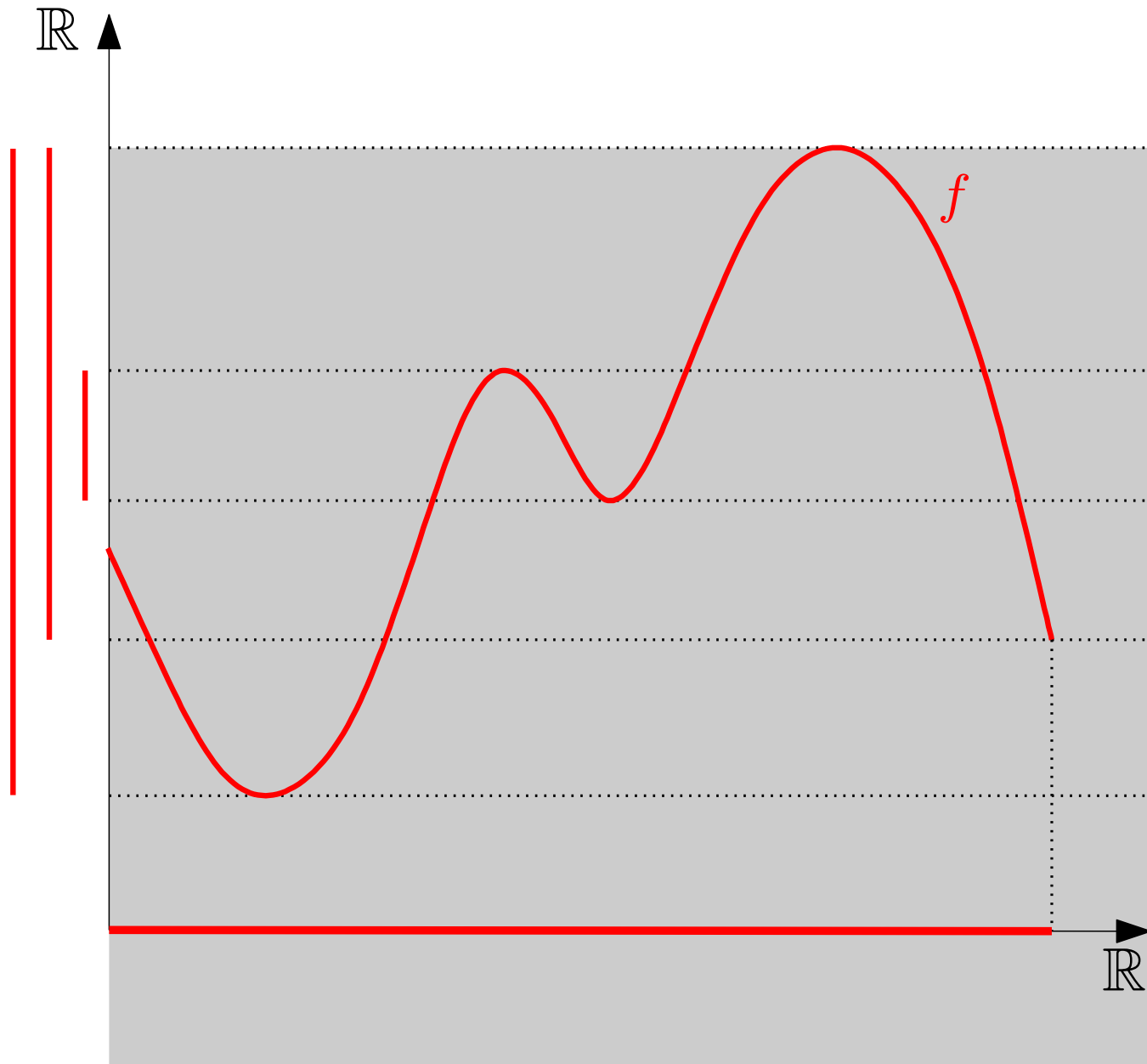- Track the evolution of the topology (homology) throughout the family
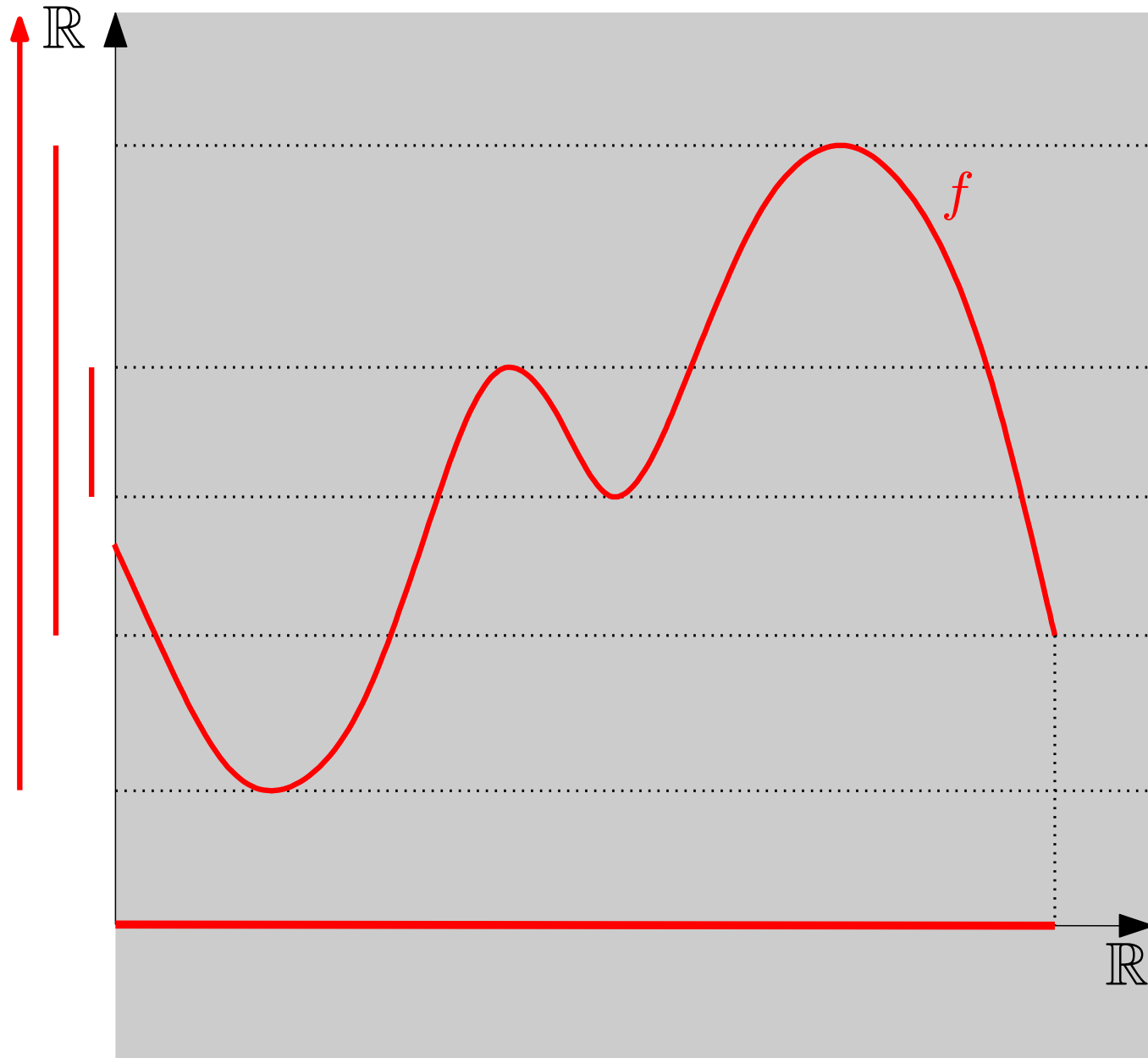
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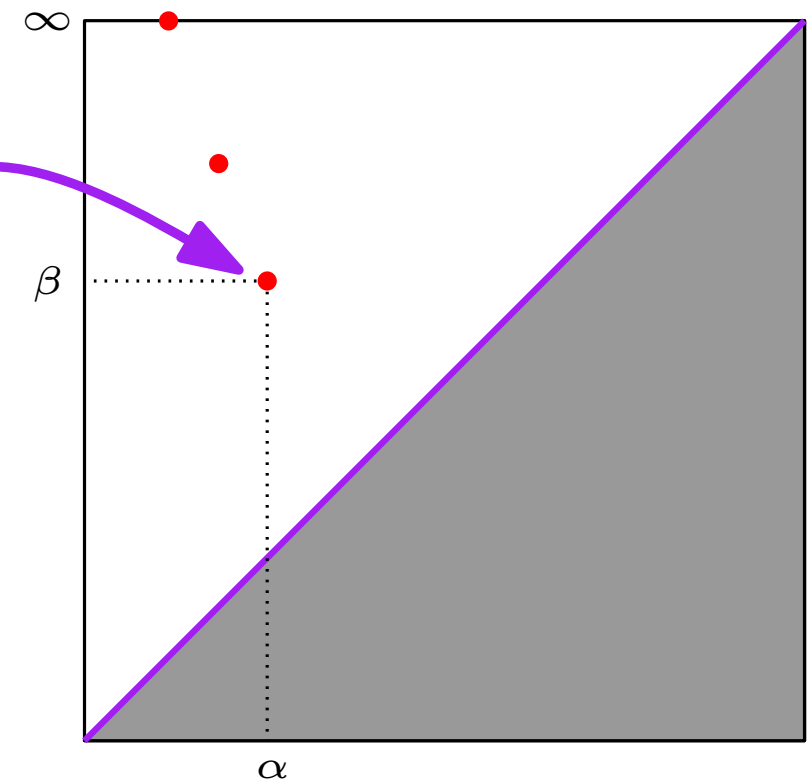- Track the evolution of the topology (homology) throughout the family

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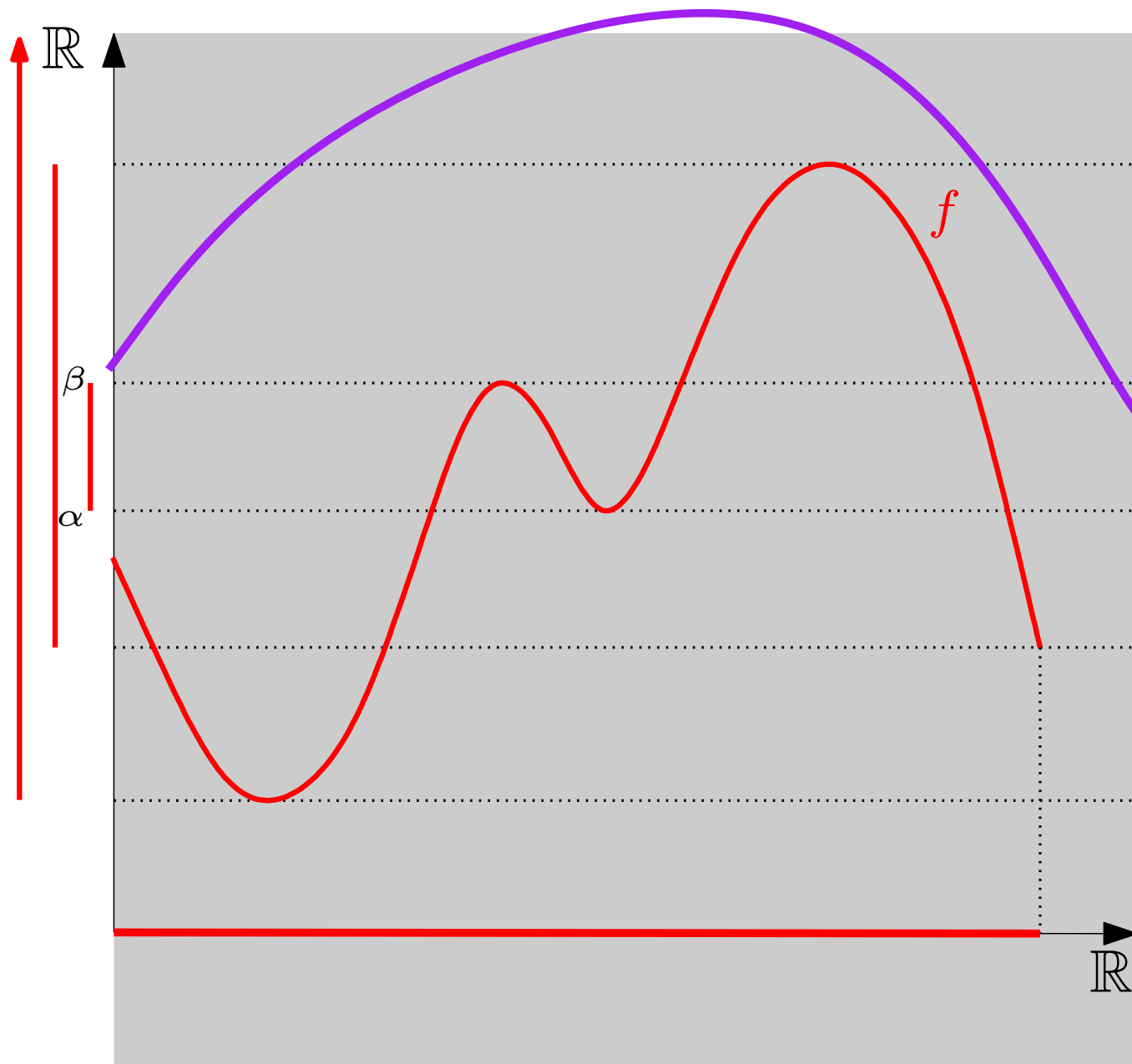
- Alternate representation as a (multi-) set of points in the plane (*diagram*).
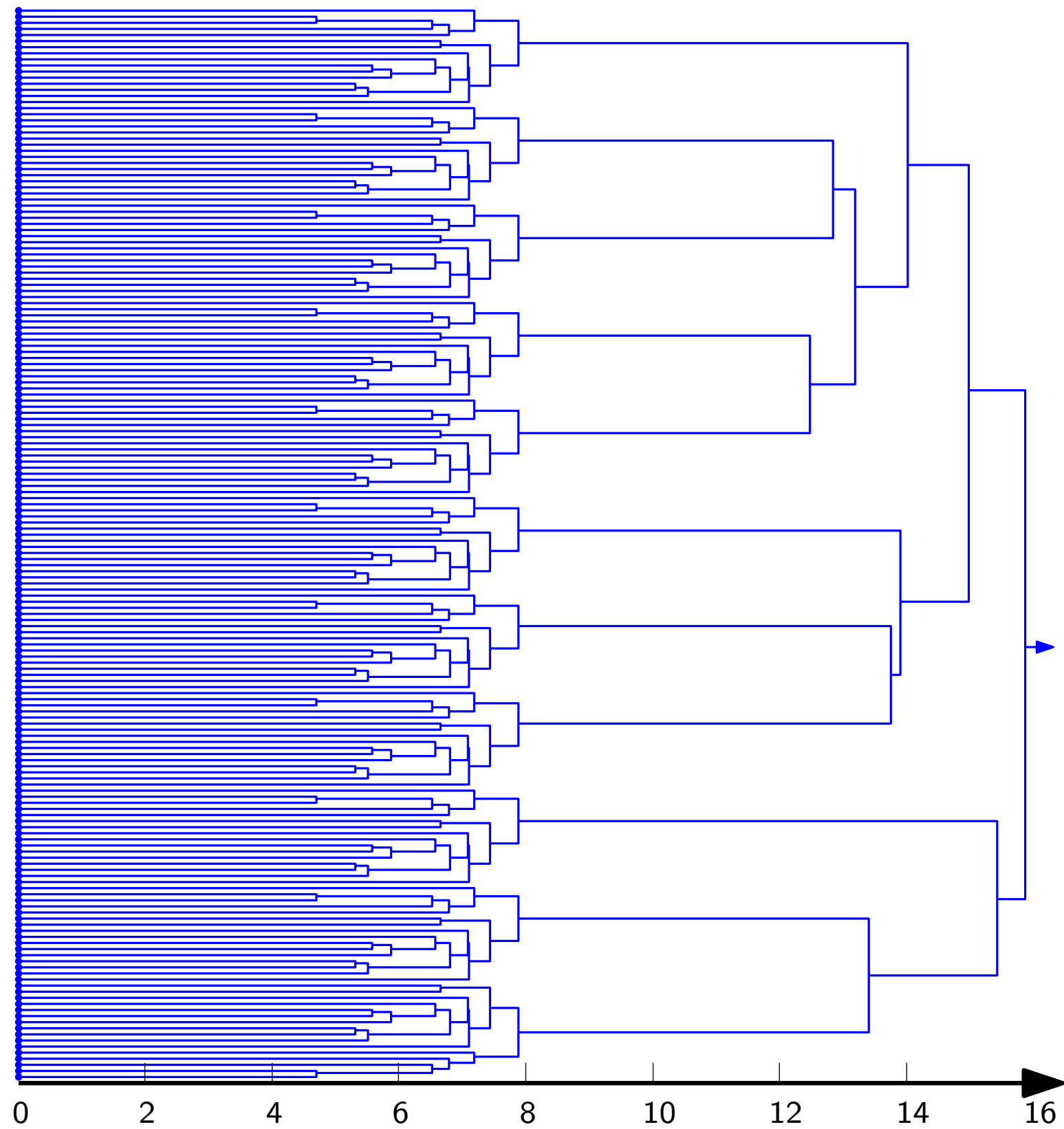
Distance Function

$$\mathrm{d}_{\mathcal{P}} : \quad \mathbb{R}^2 \to \mathbb{R}$$
$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$

scale

3

Distance Function

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3

Distance Function

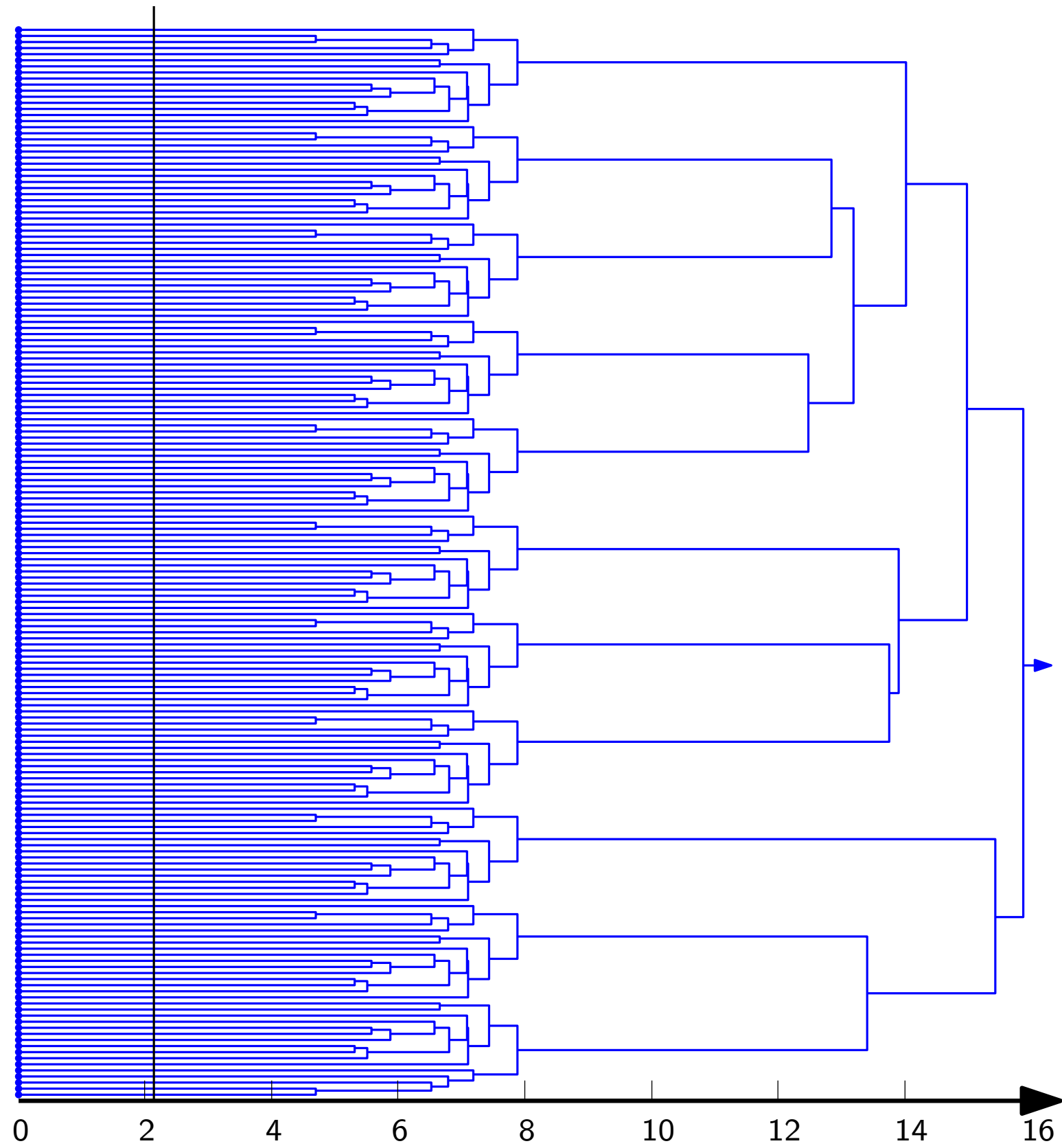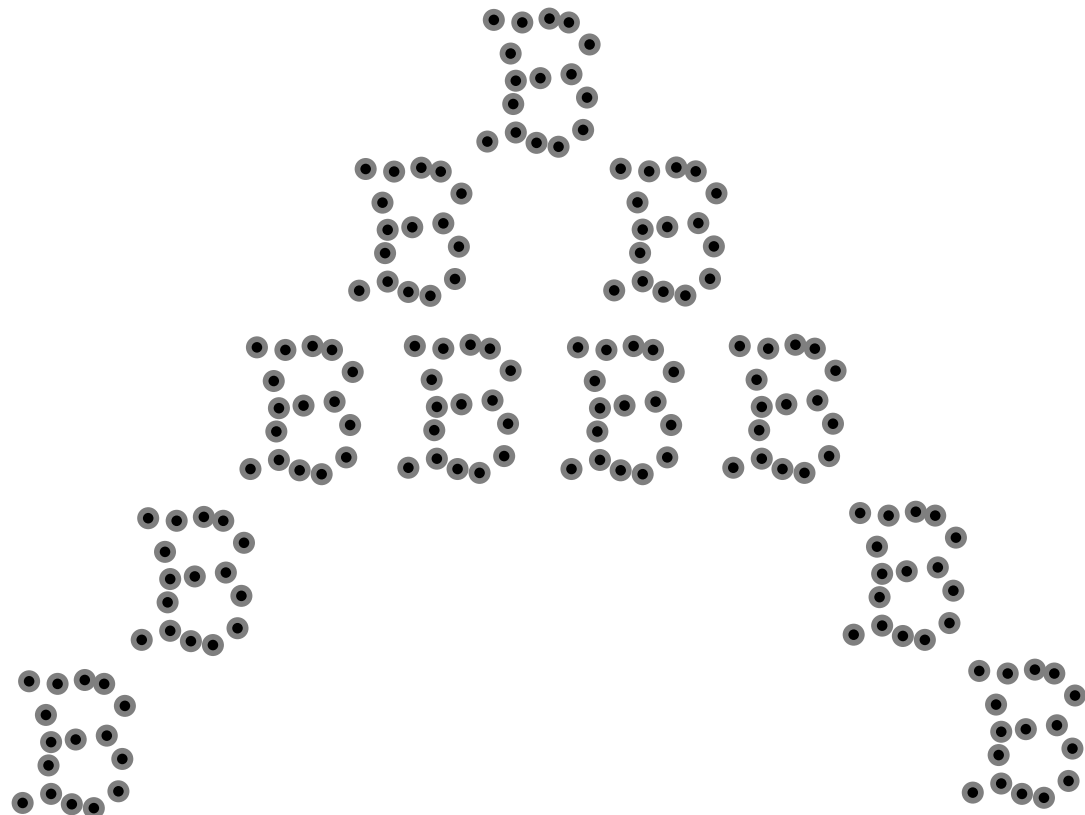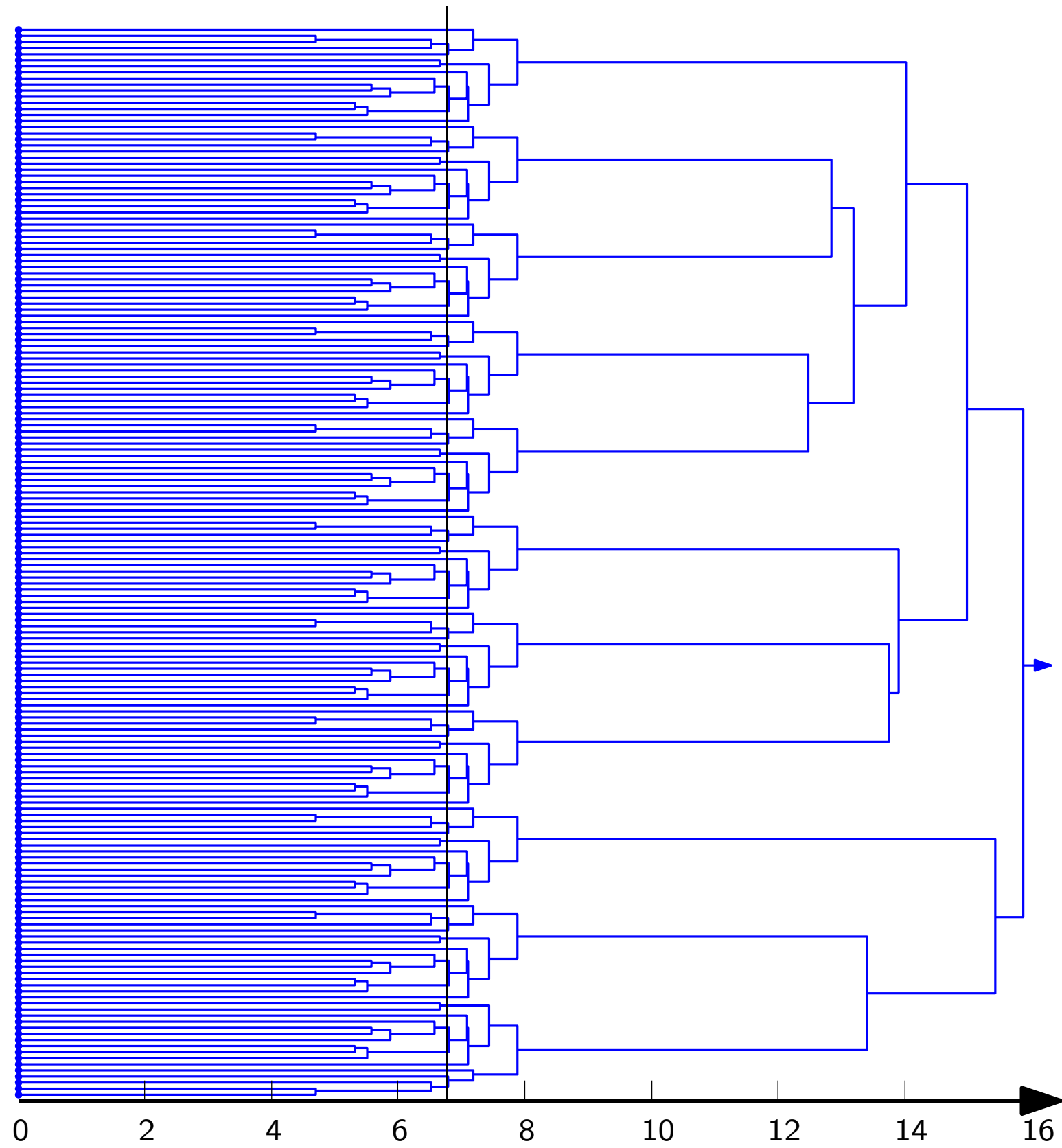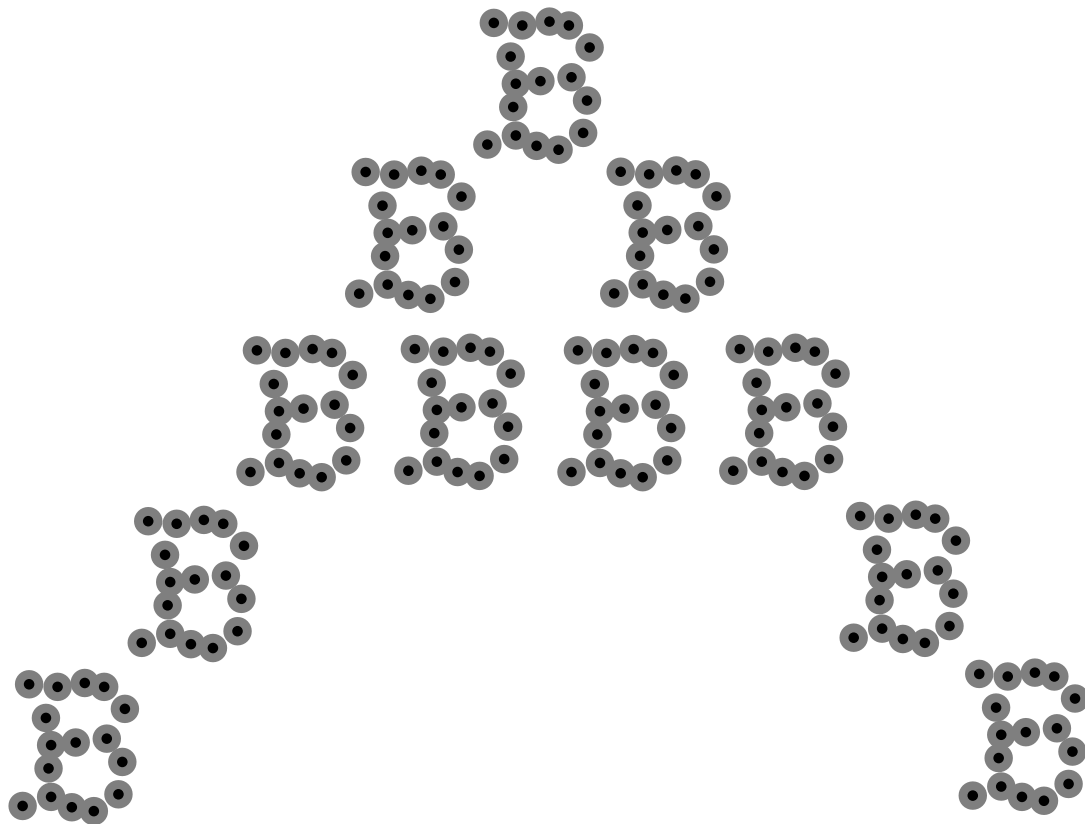$$\mathrm{d}_{\mathcal{P}} : \quad \mathbb{R}^2 \to \mathbb{R}$$

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Distance Function

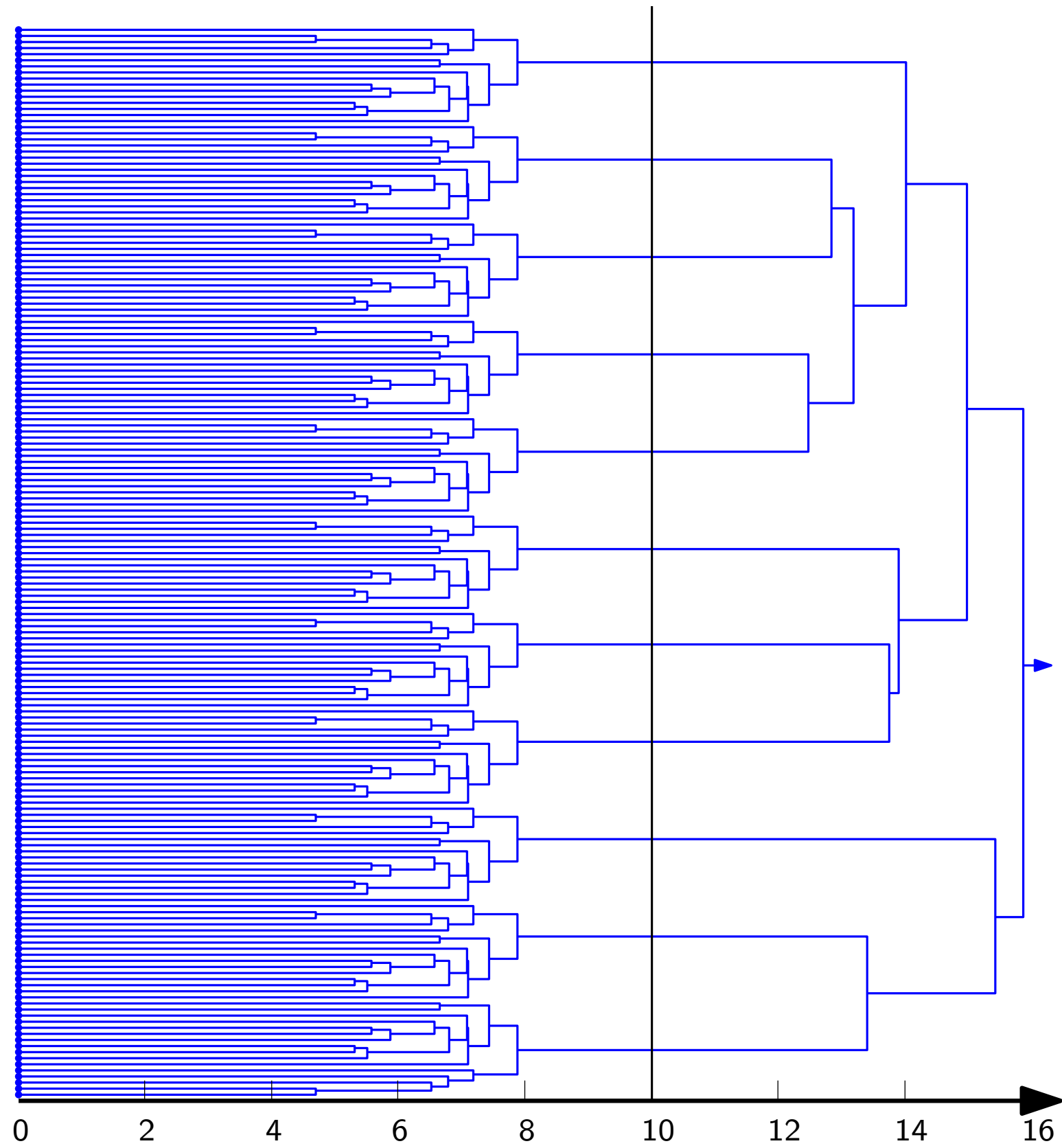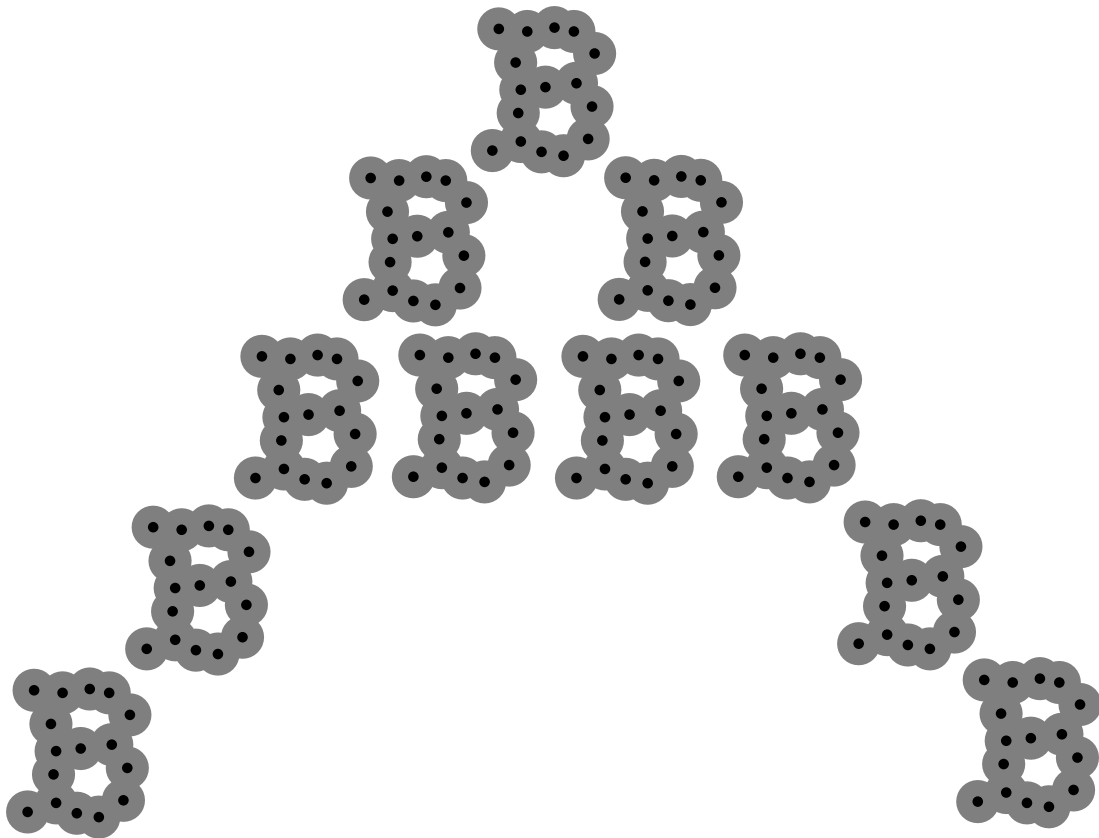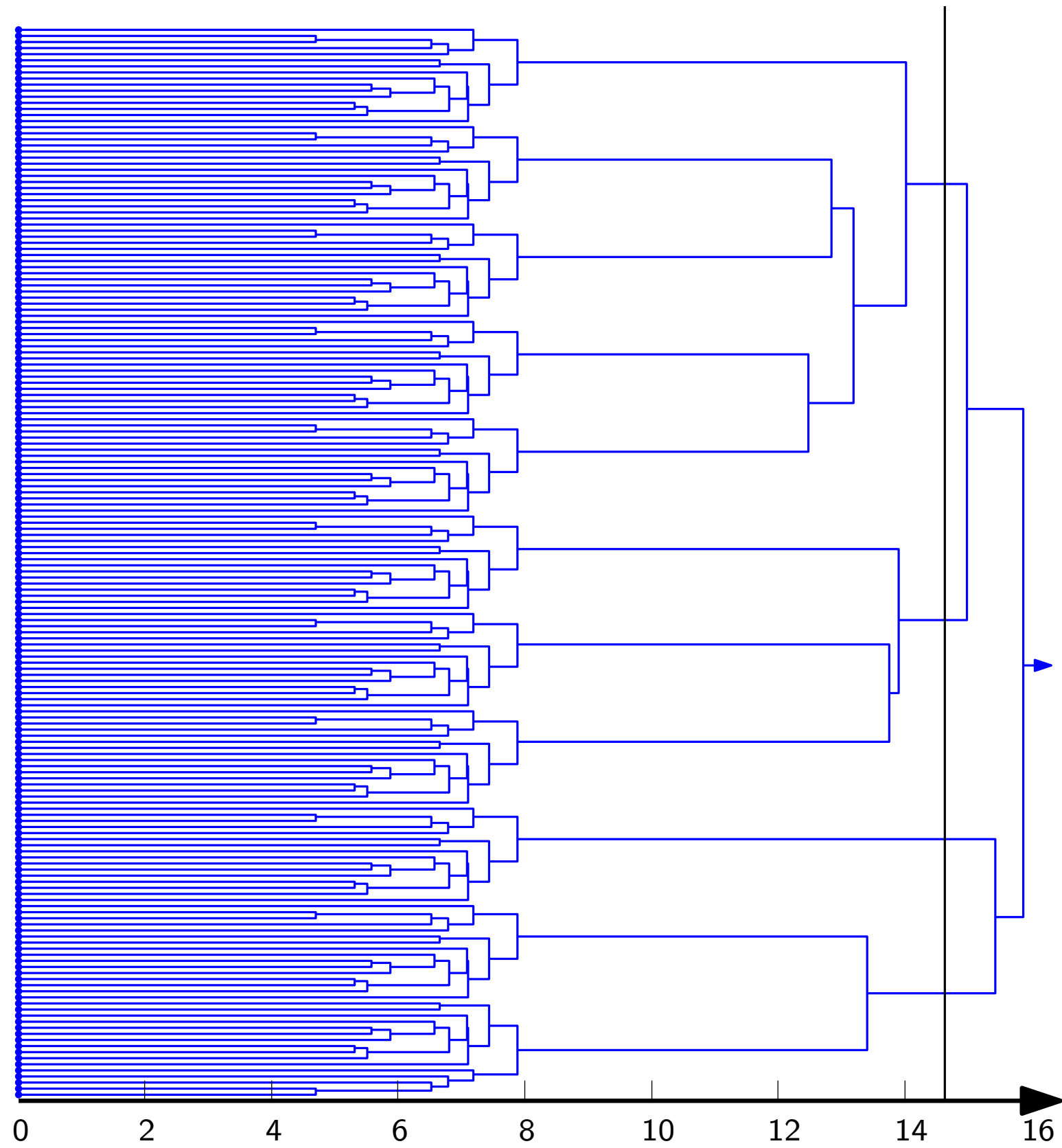$$\mathrm{d}_{\mathcal{P}} : \quad \mathbb{R}^2 \to \mathbb{R}$$
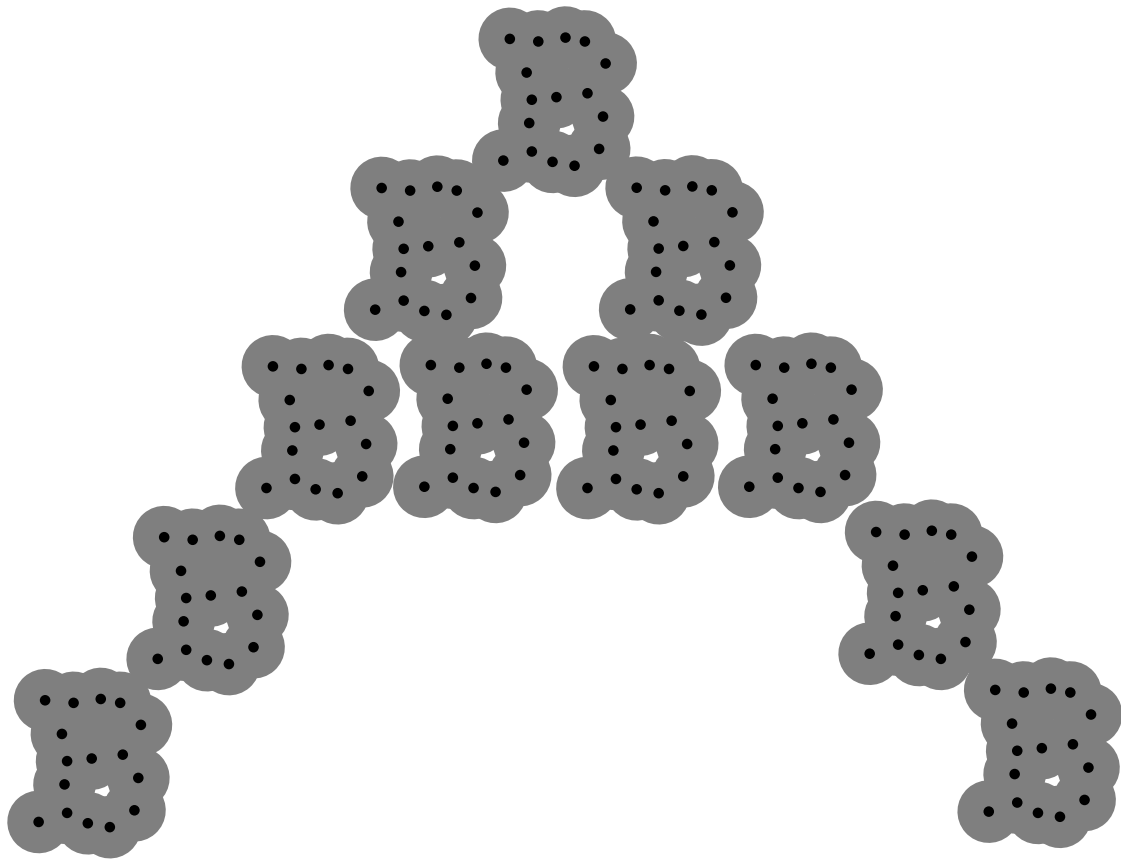$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$

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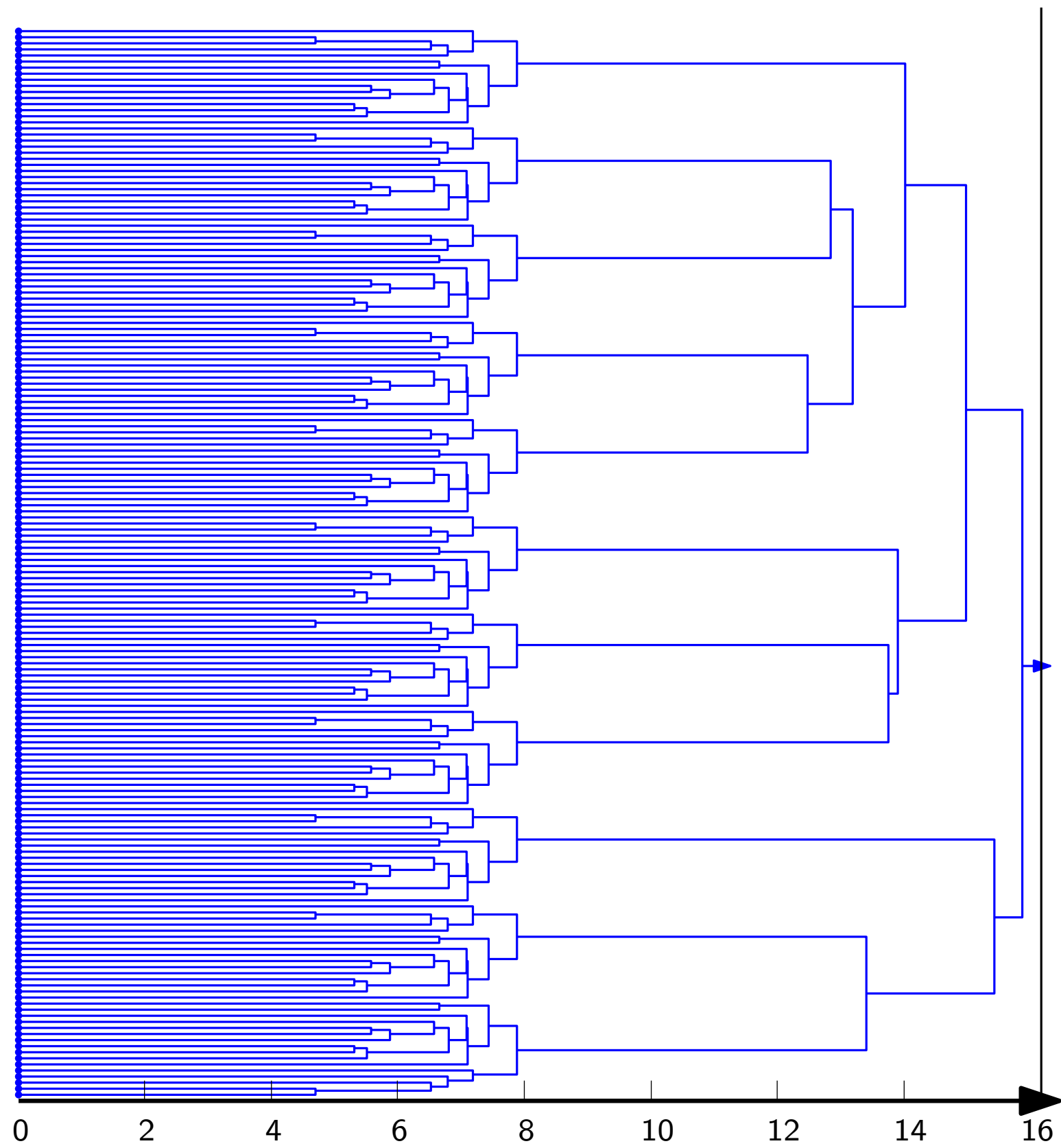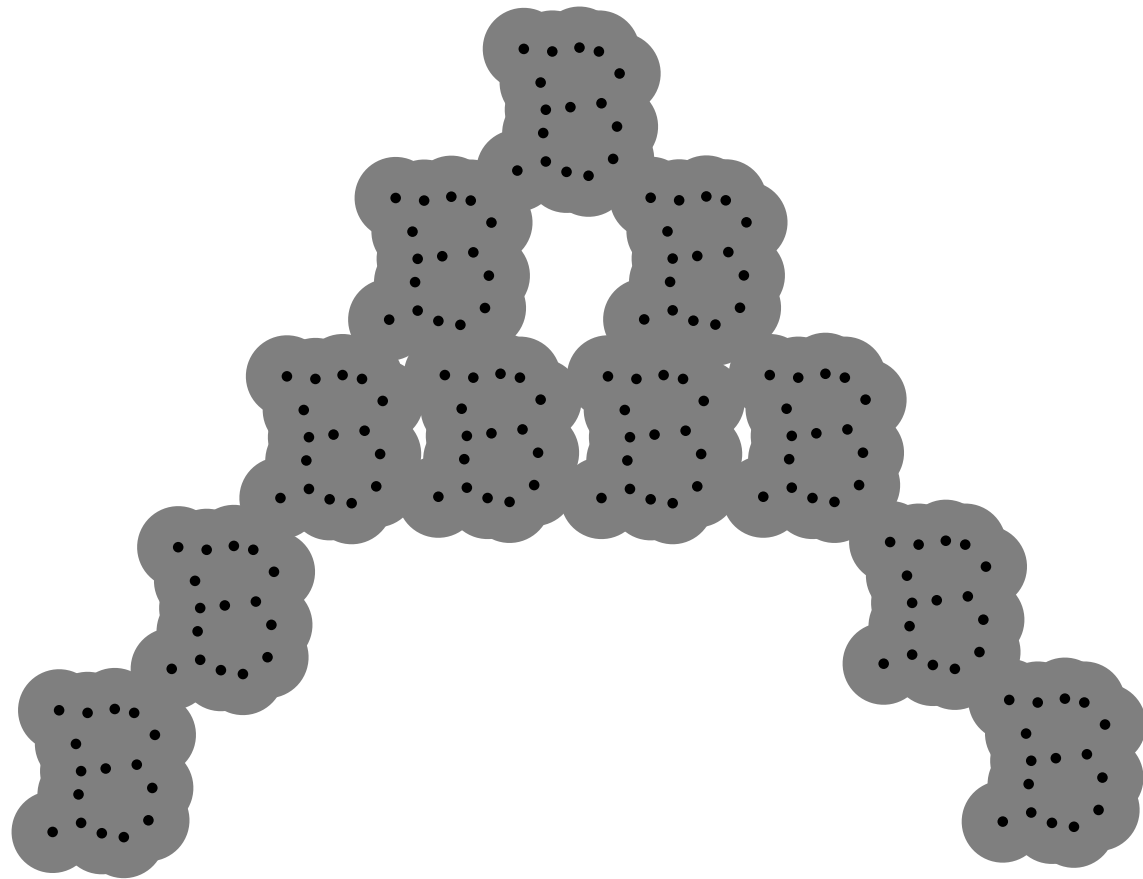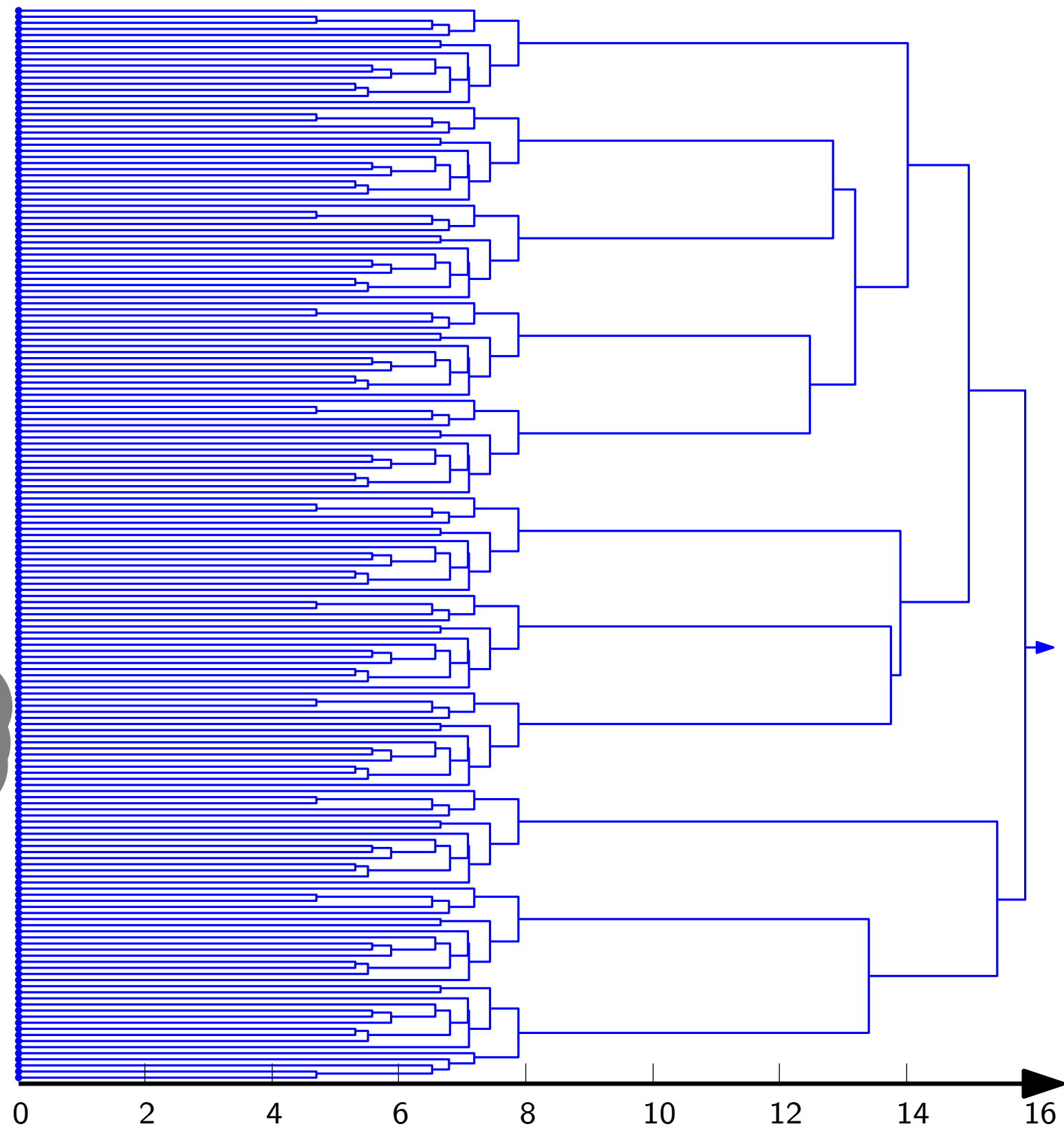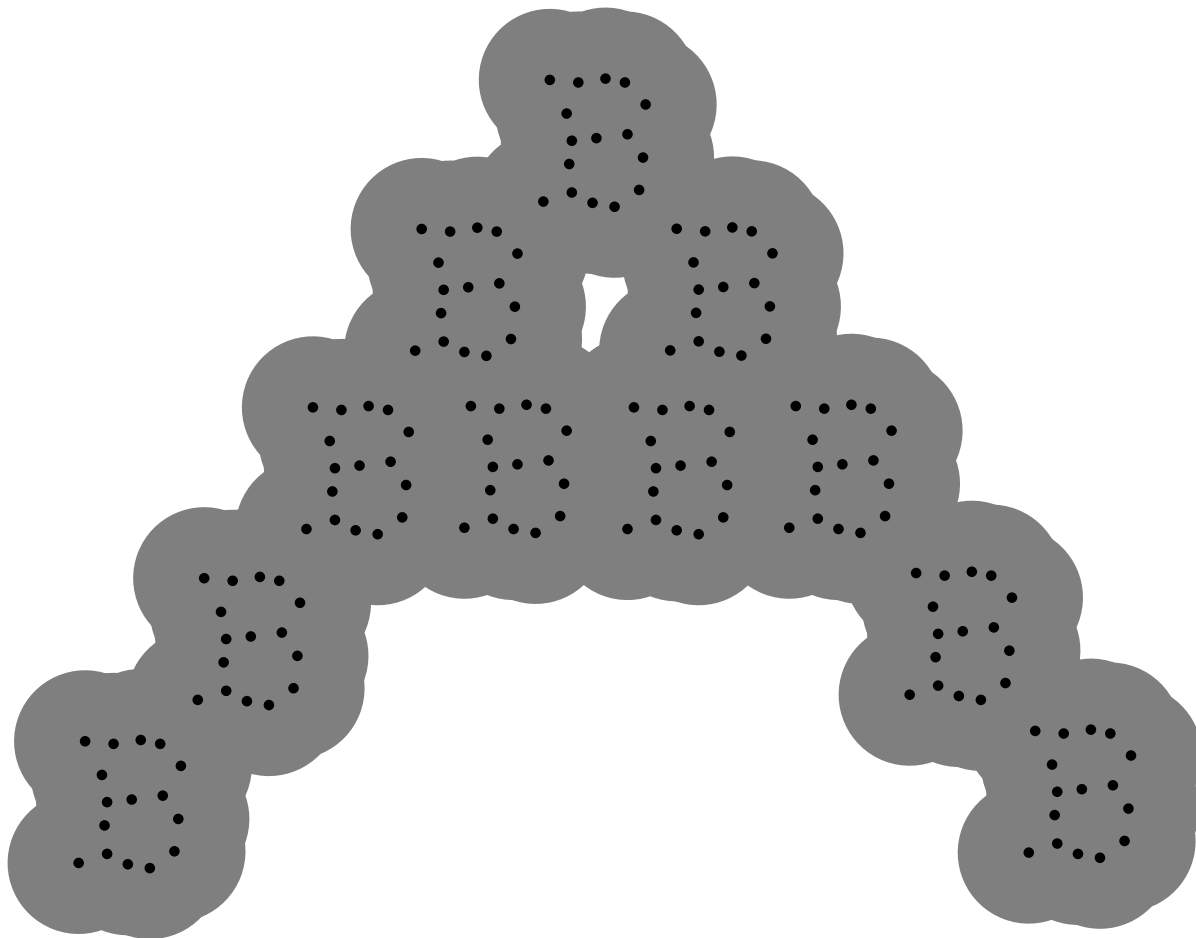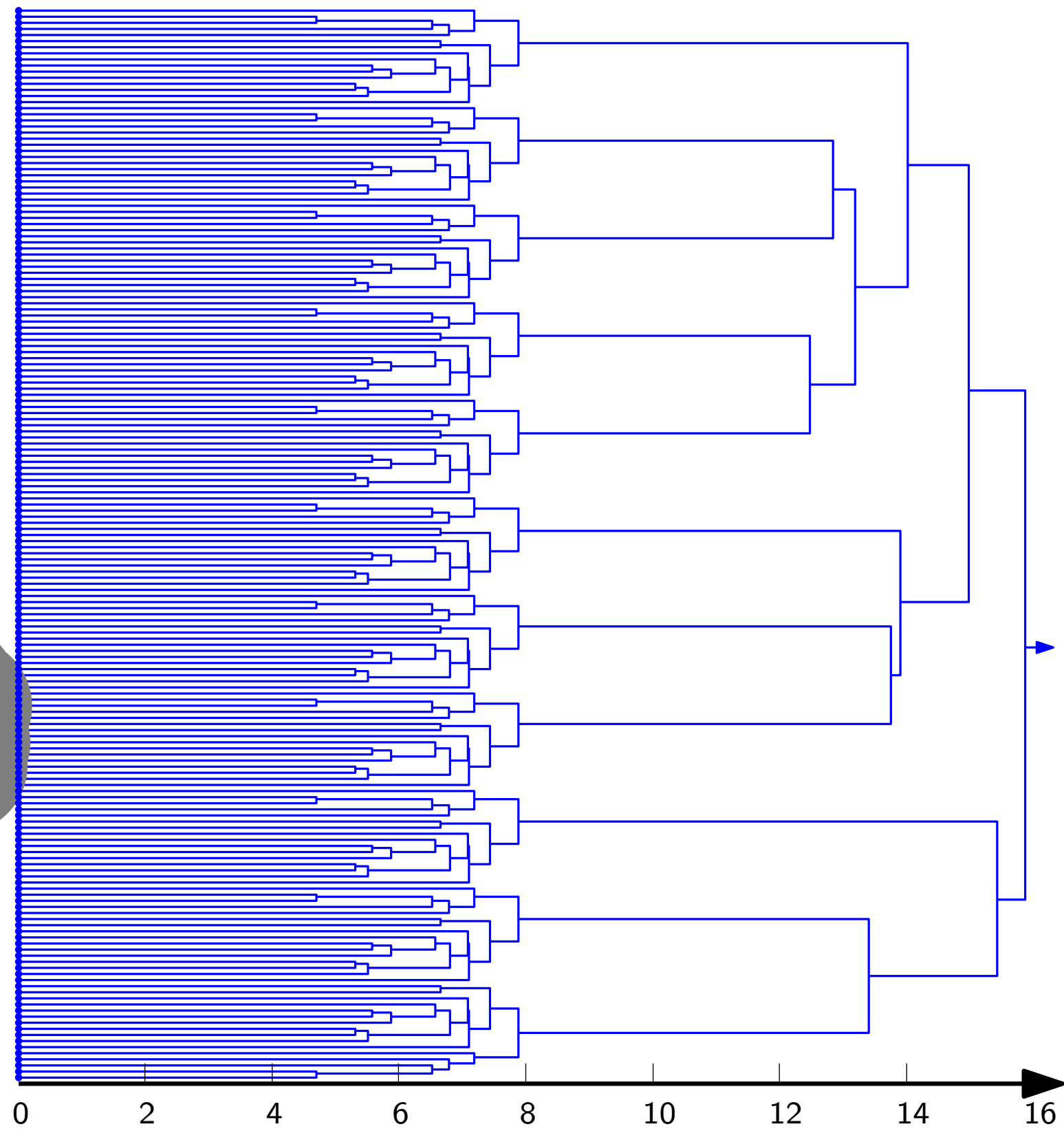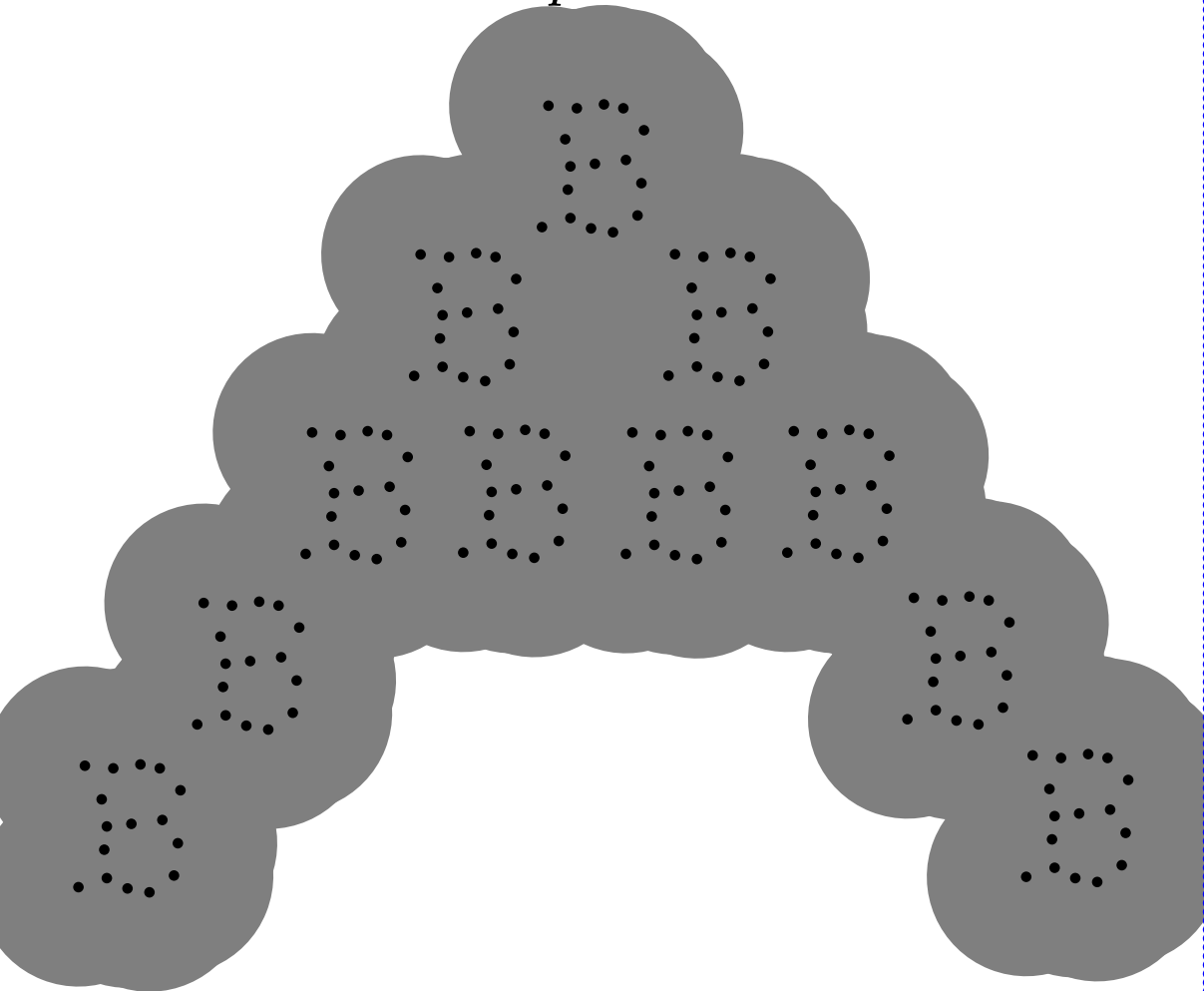$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$

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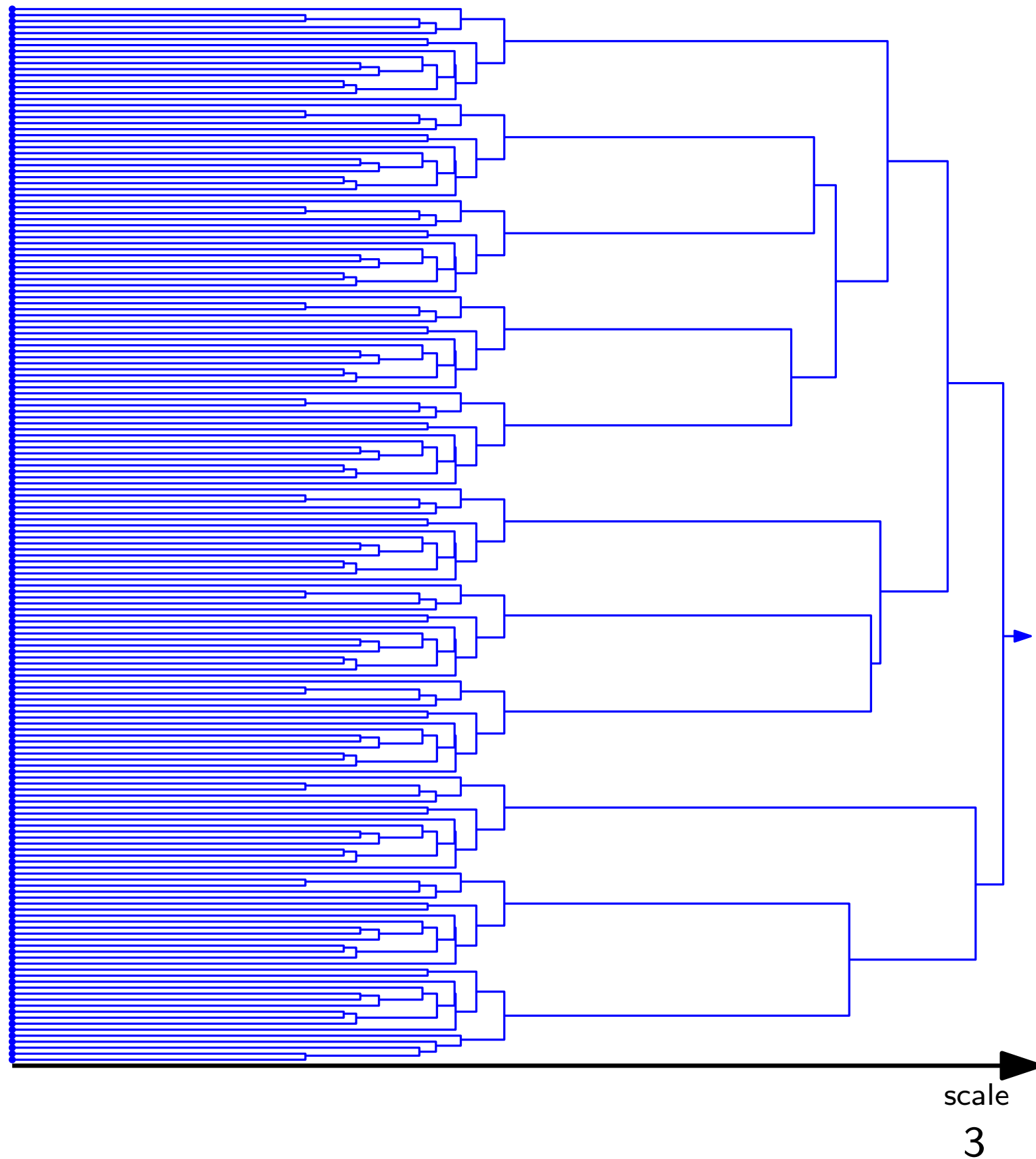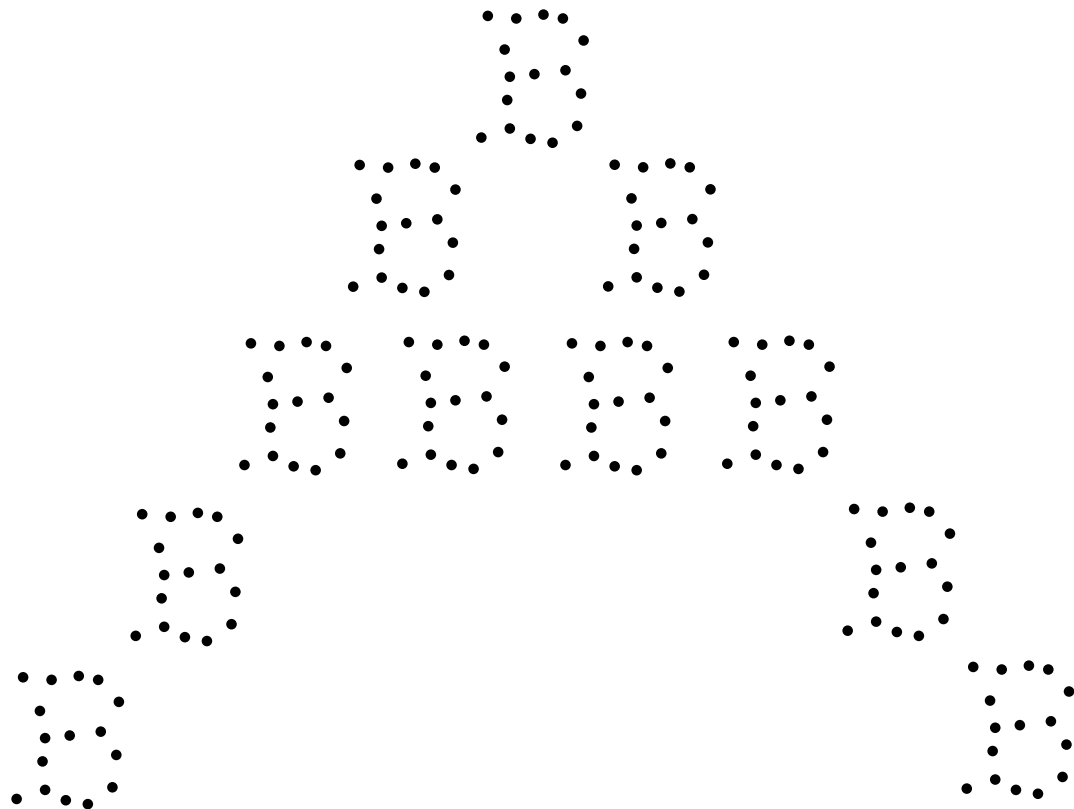
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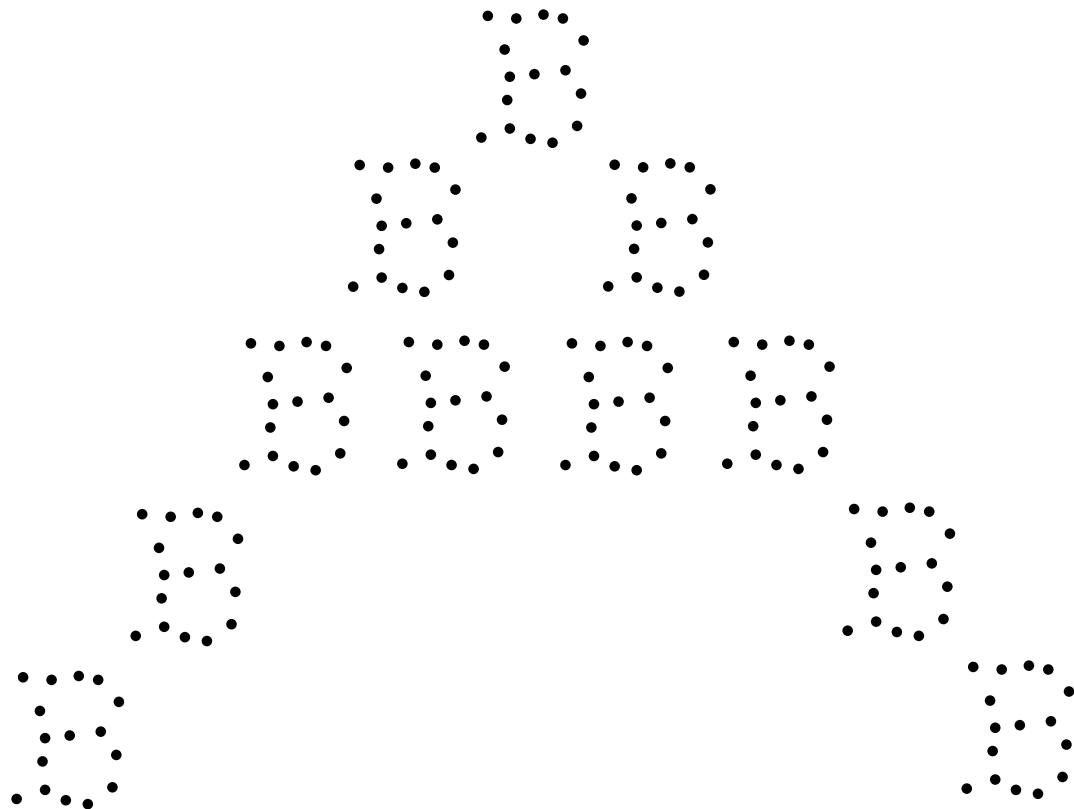
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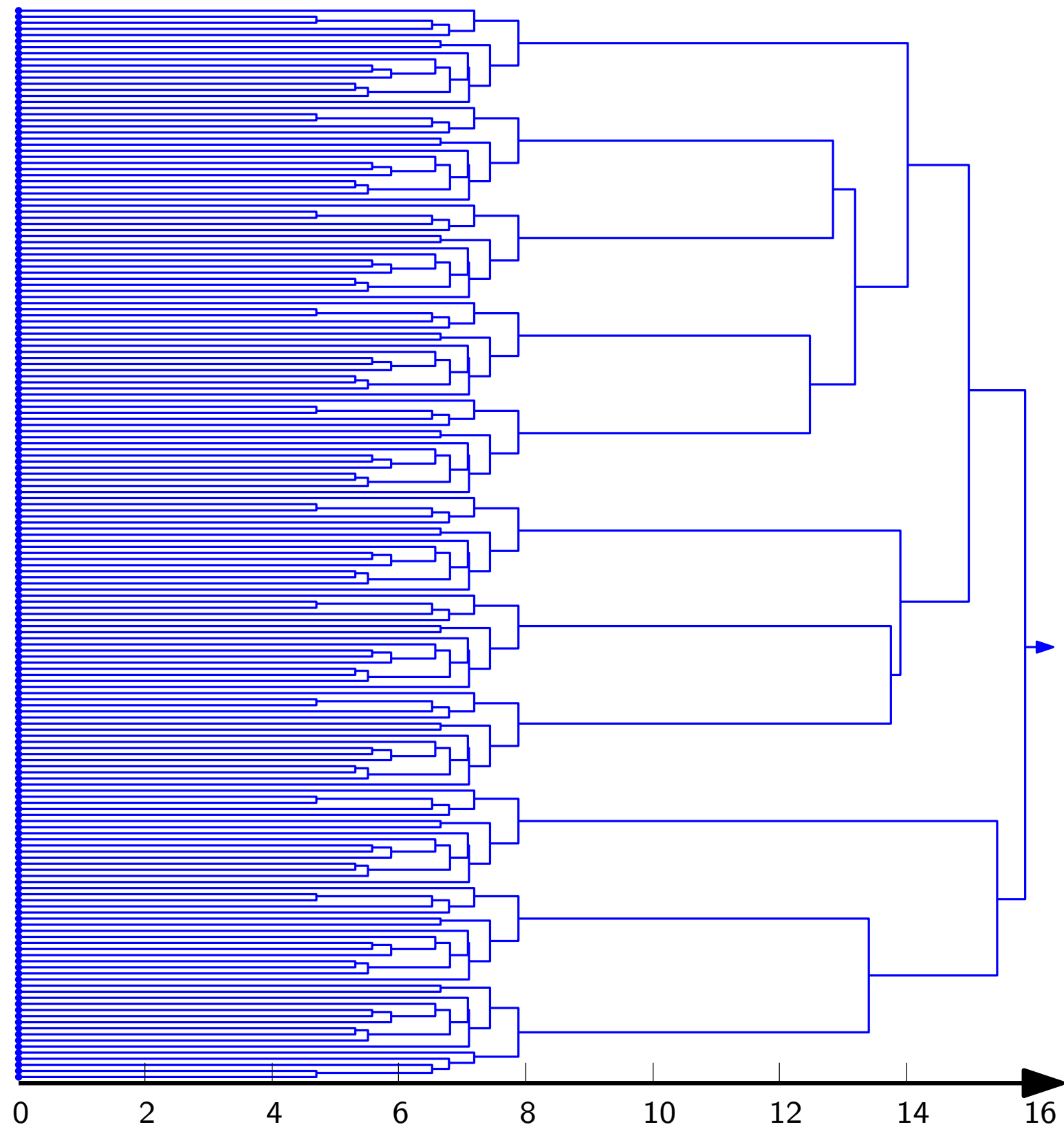

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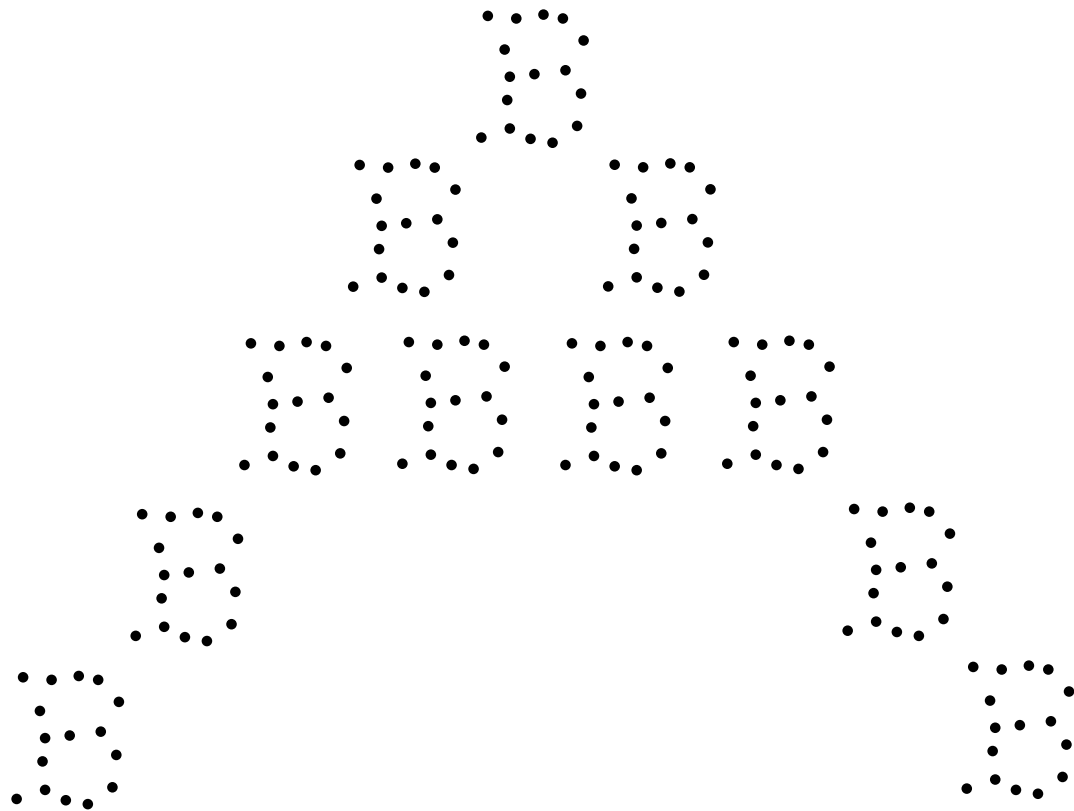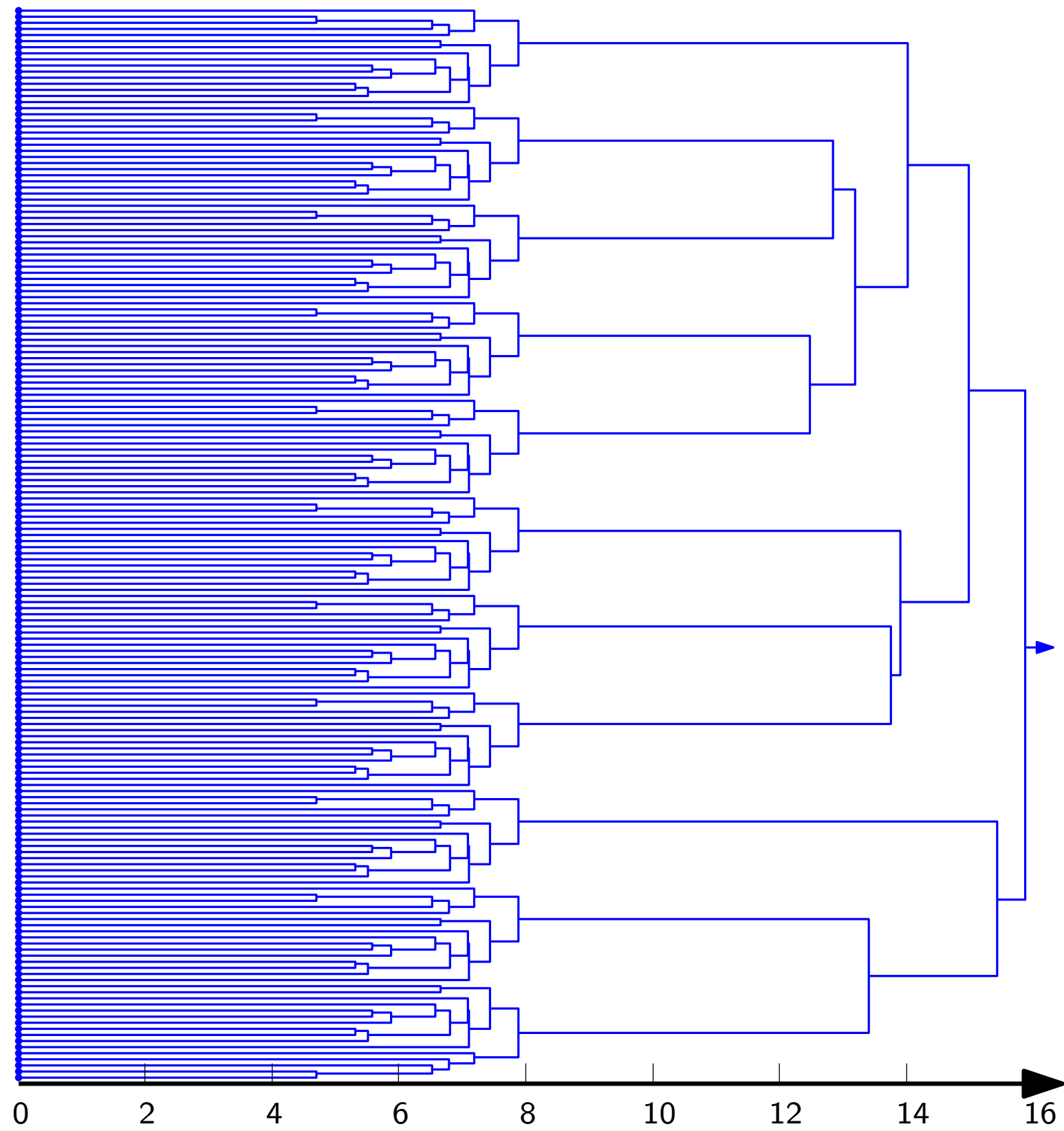
dendrogram is:

- informative
- unstable

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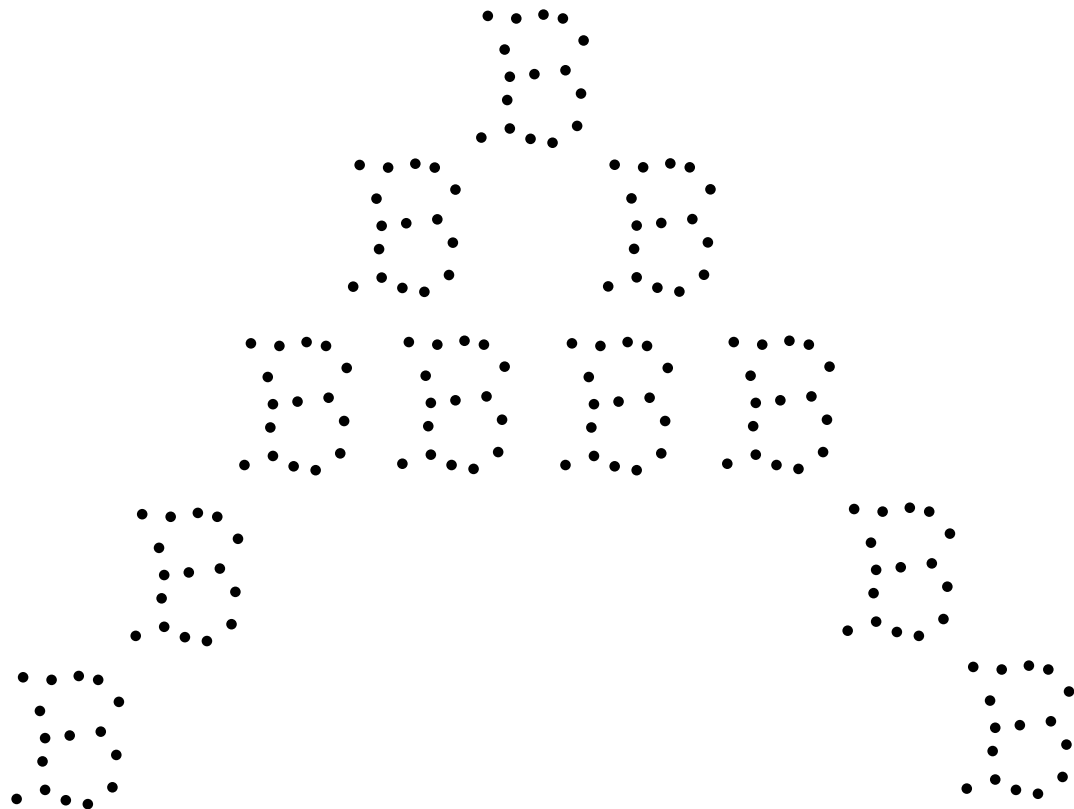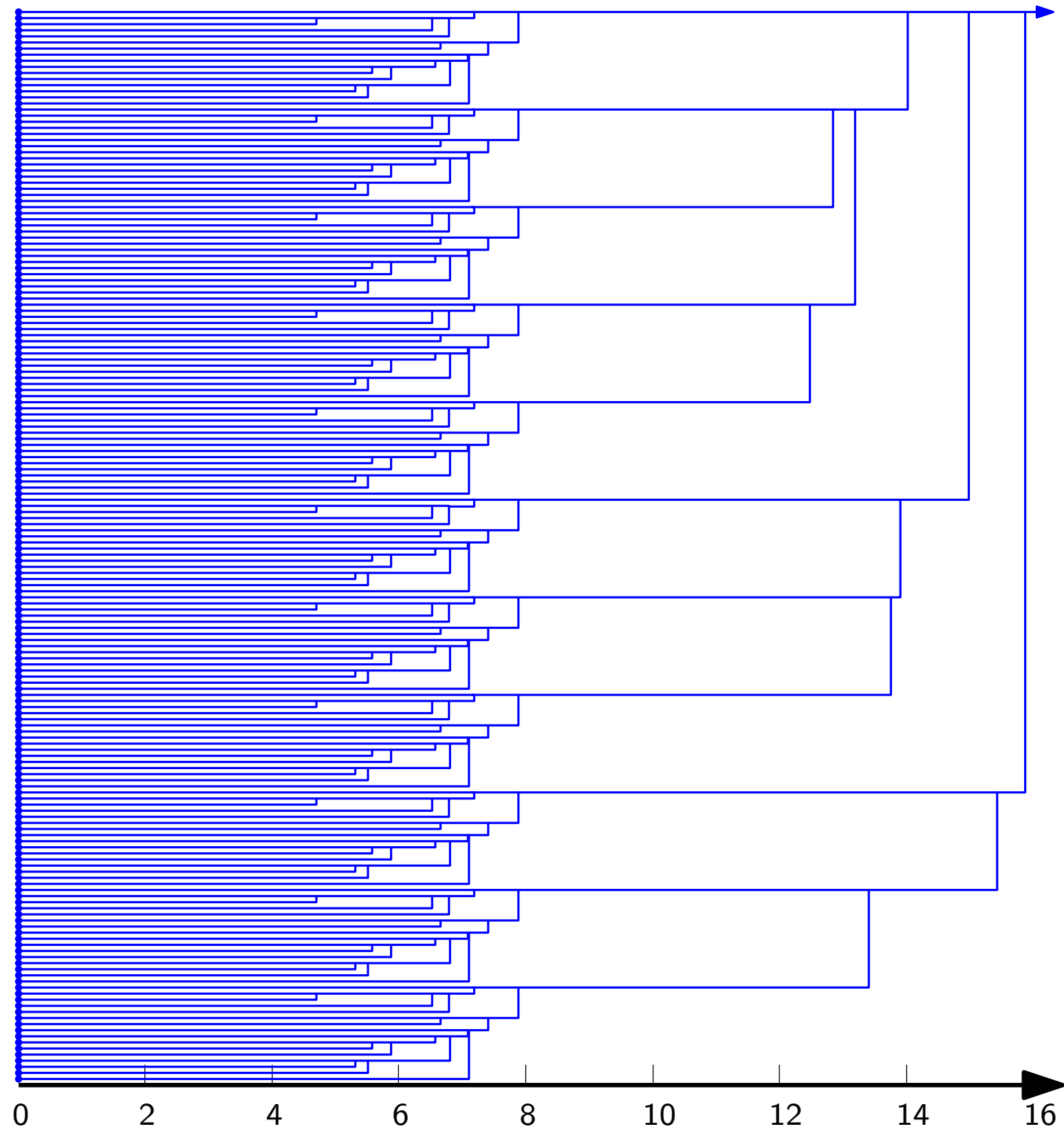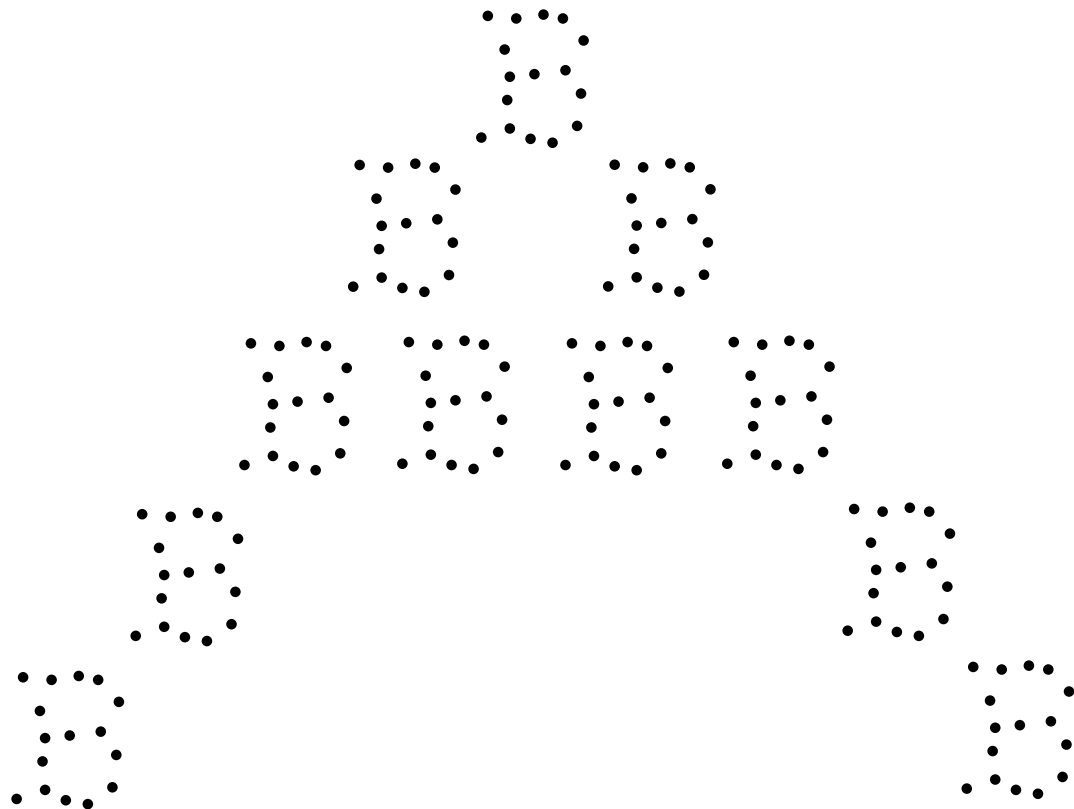$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$

dendrogram $\to$ barcode

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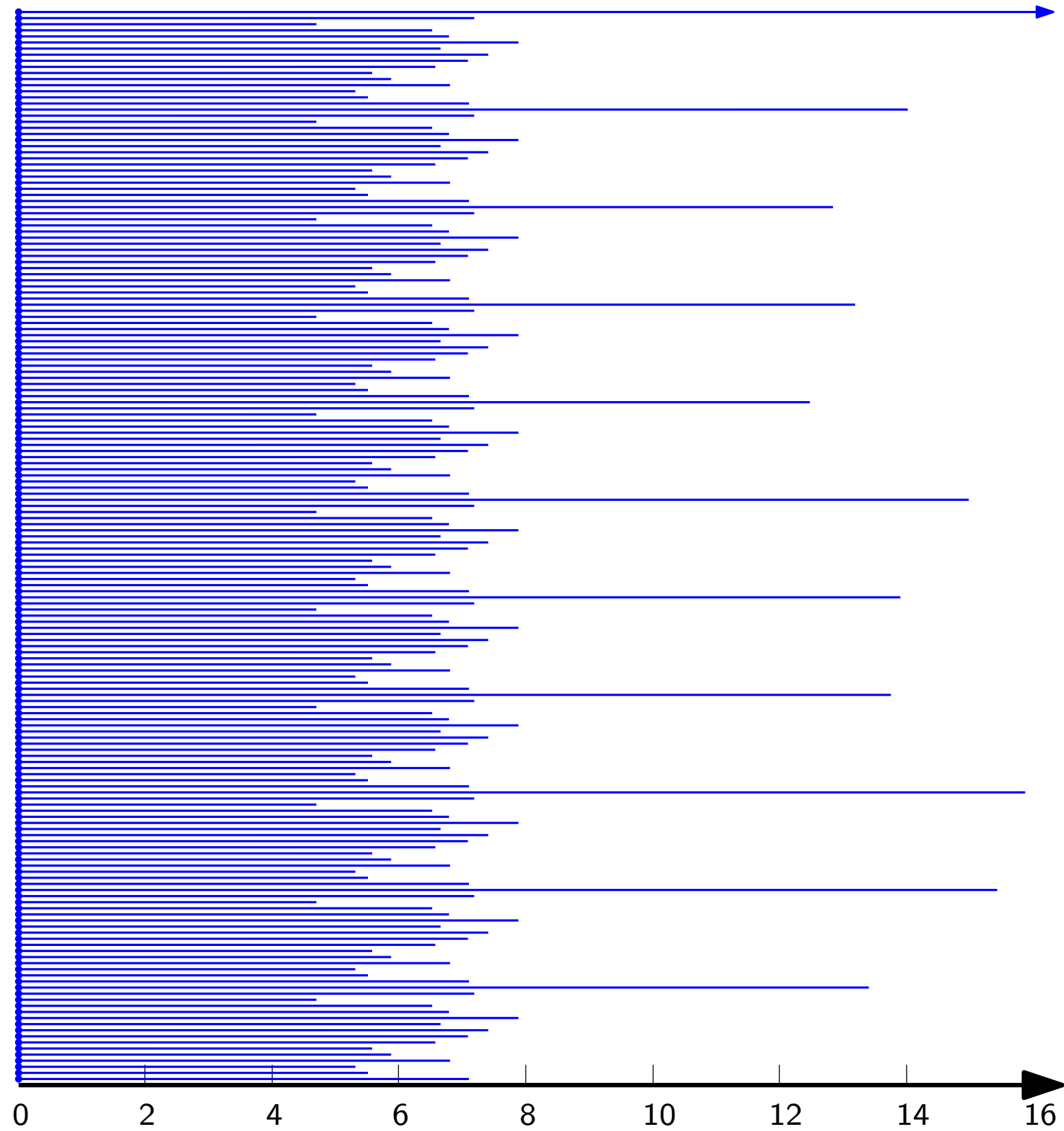
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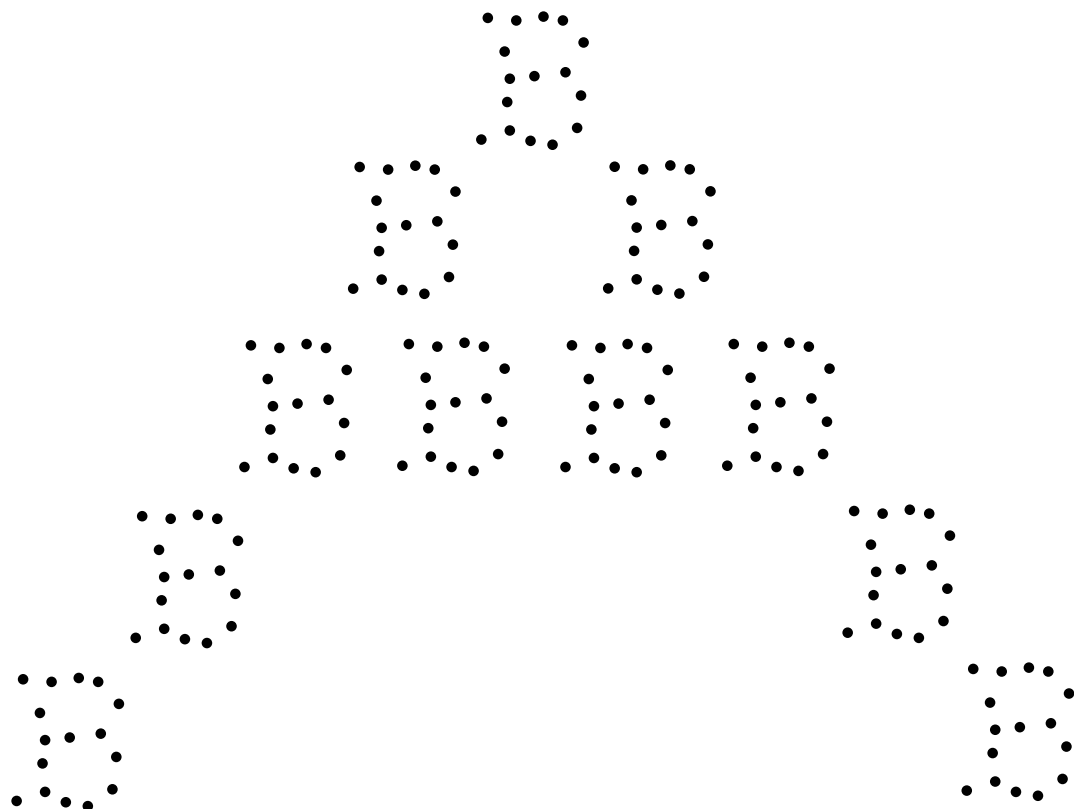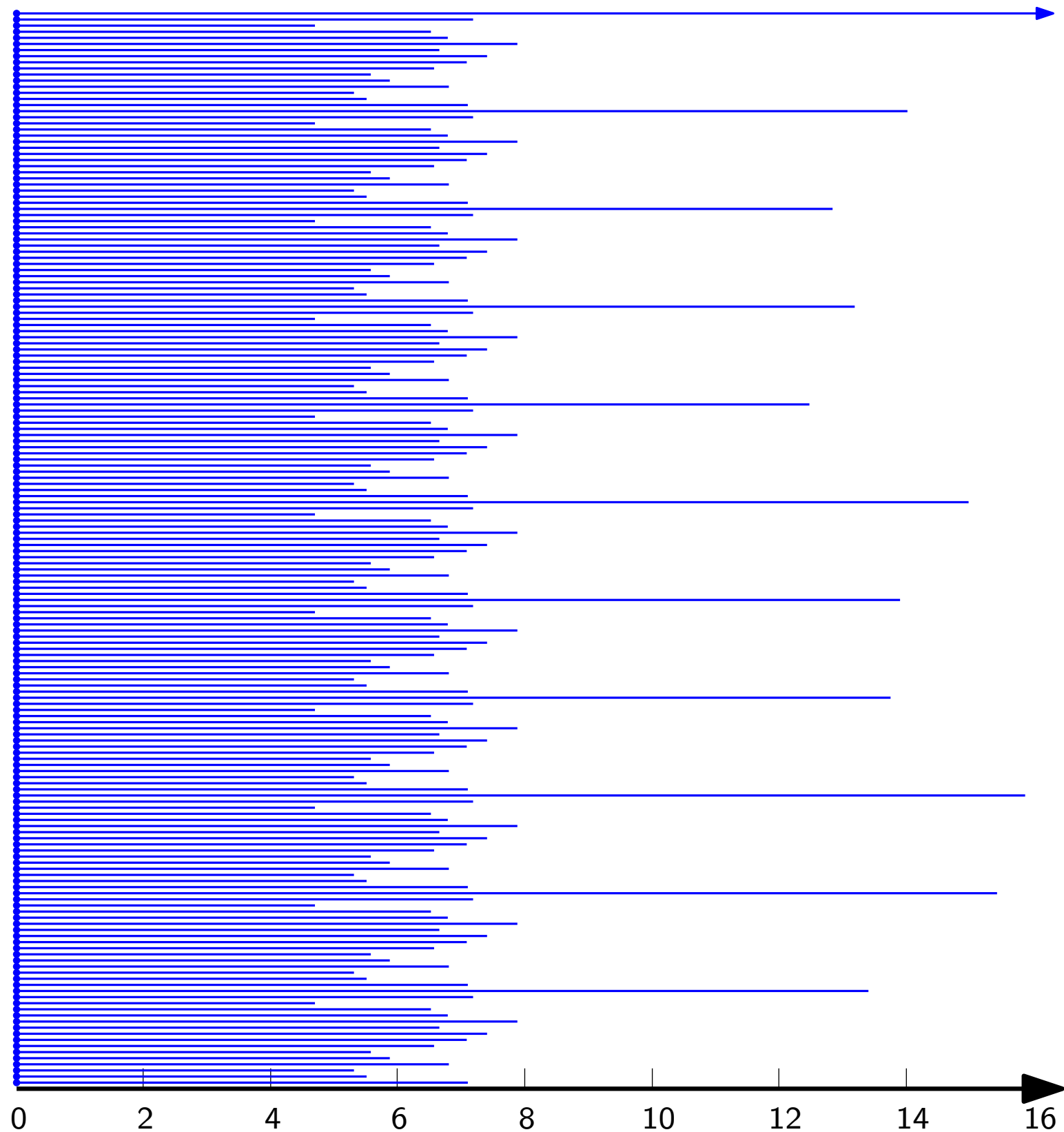
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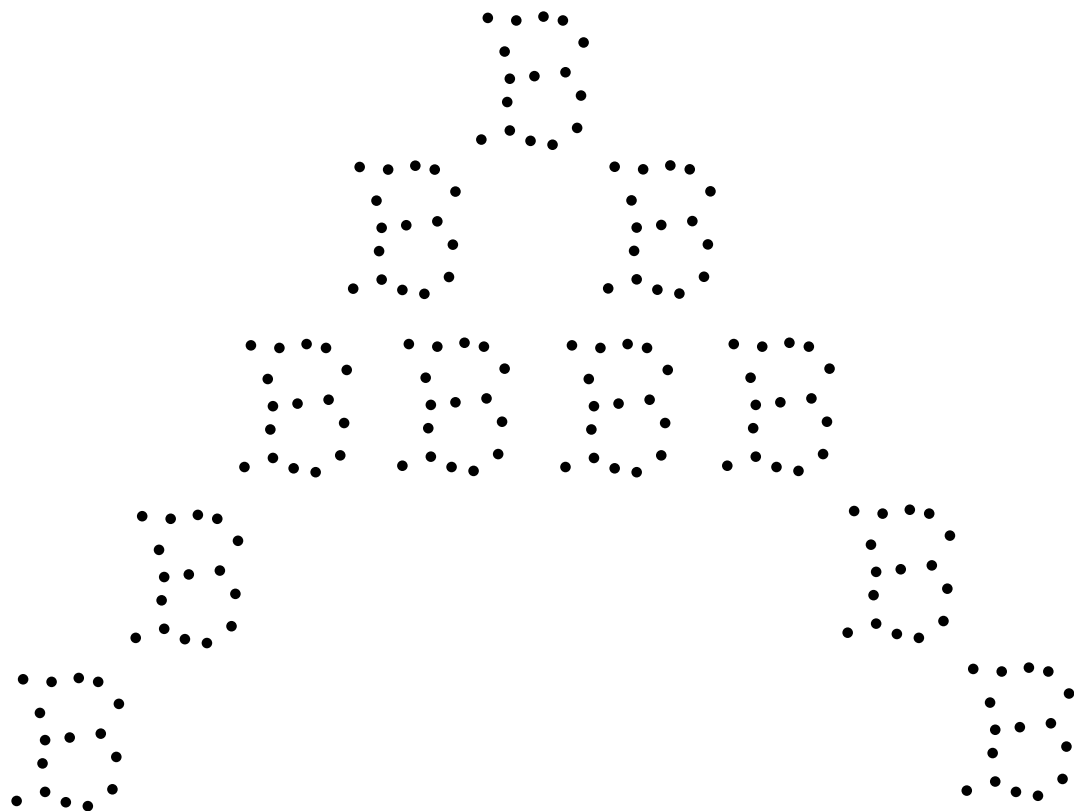

barcode is:

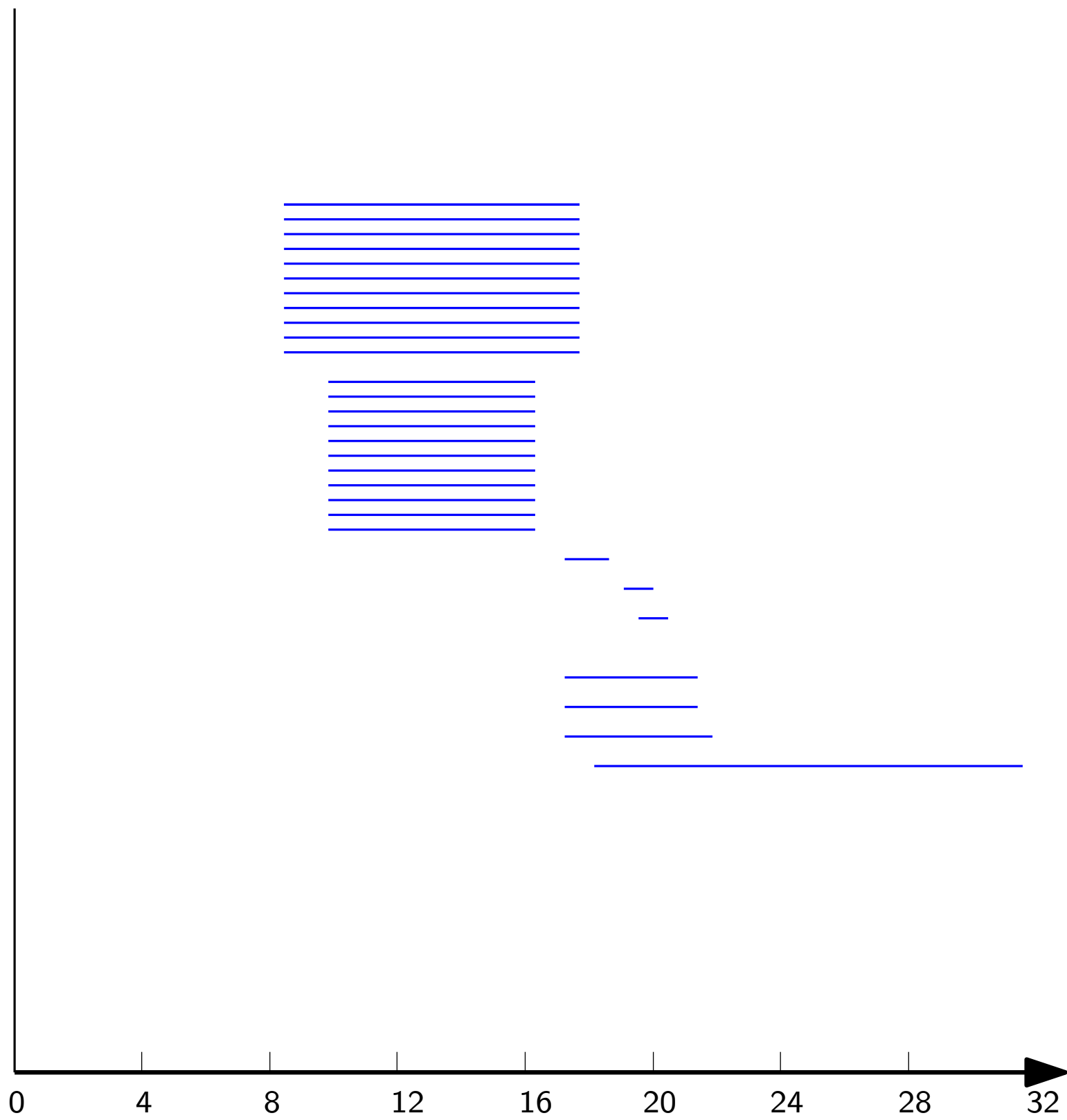- less (but still) informative
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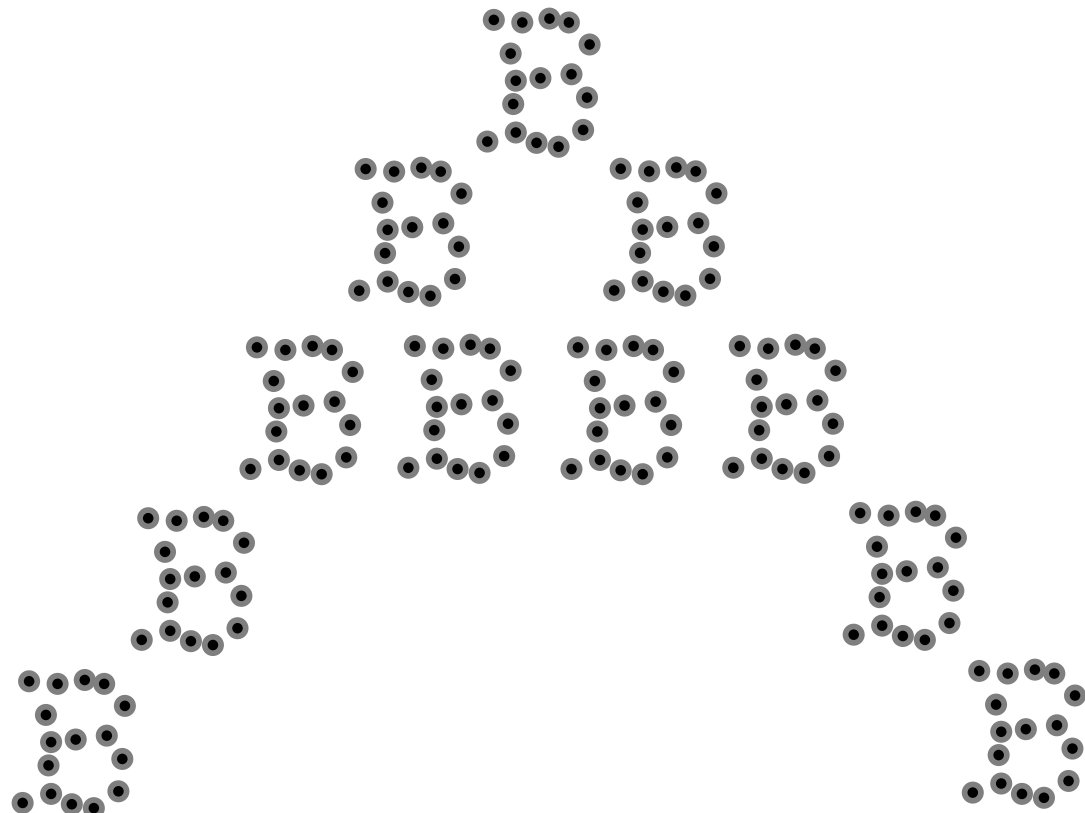
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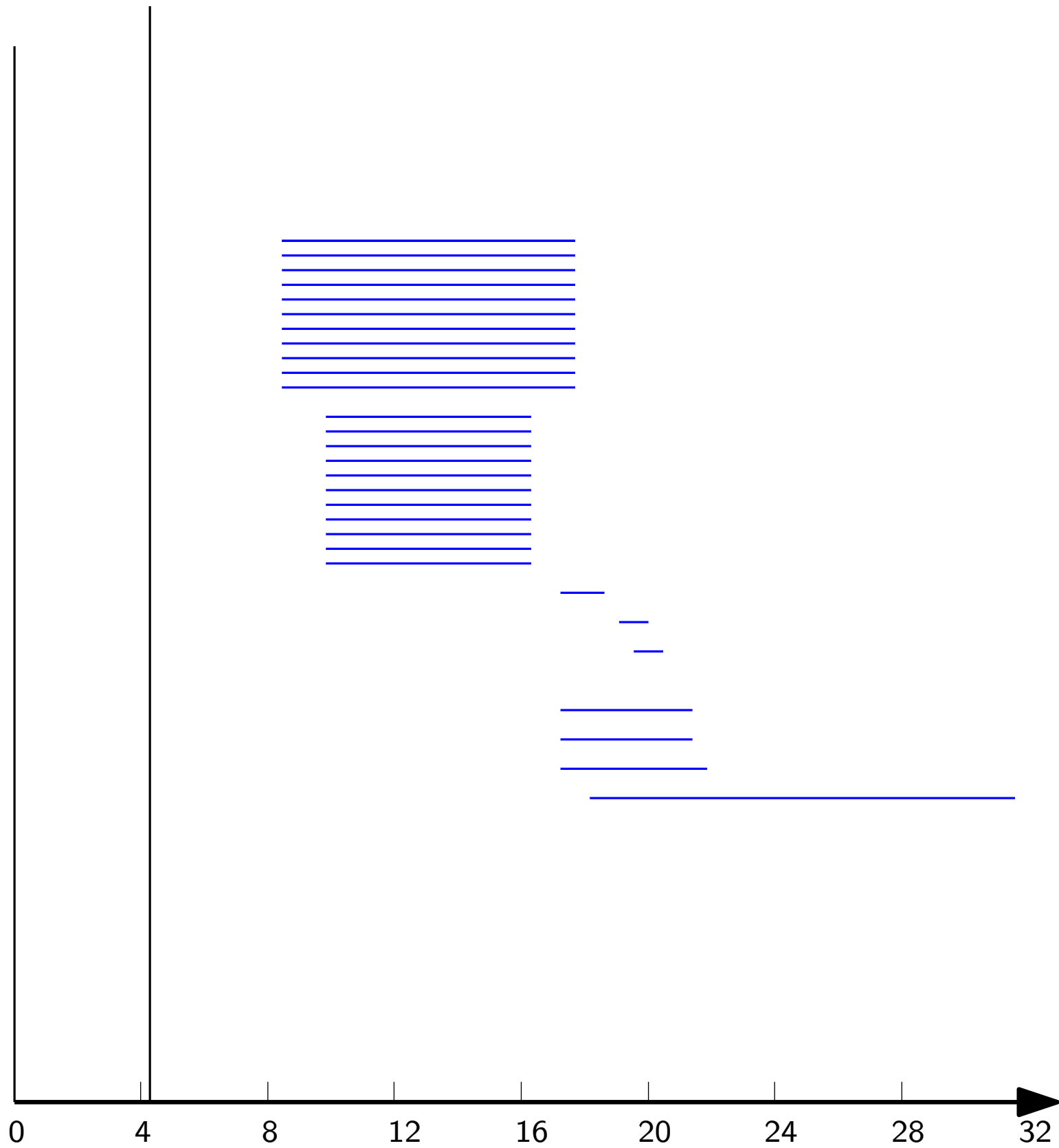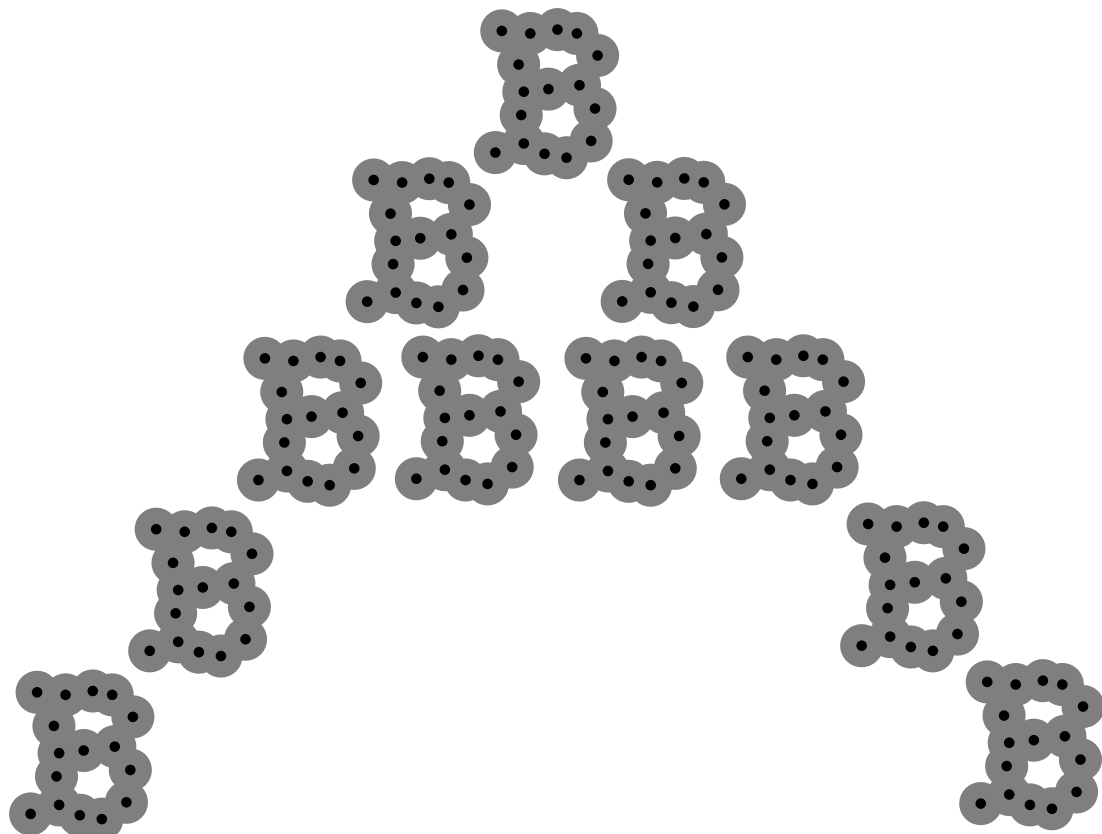- generalizable

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0   4   8   12   16   20   24   28   32

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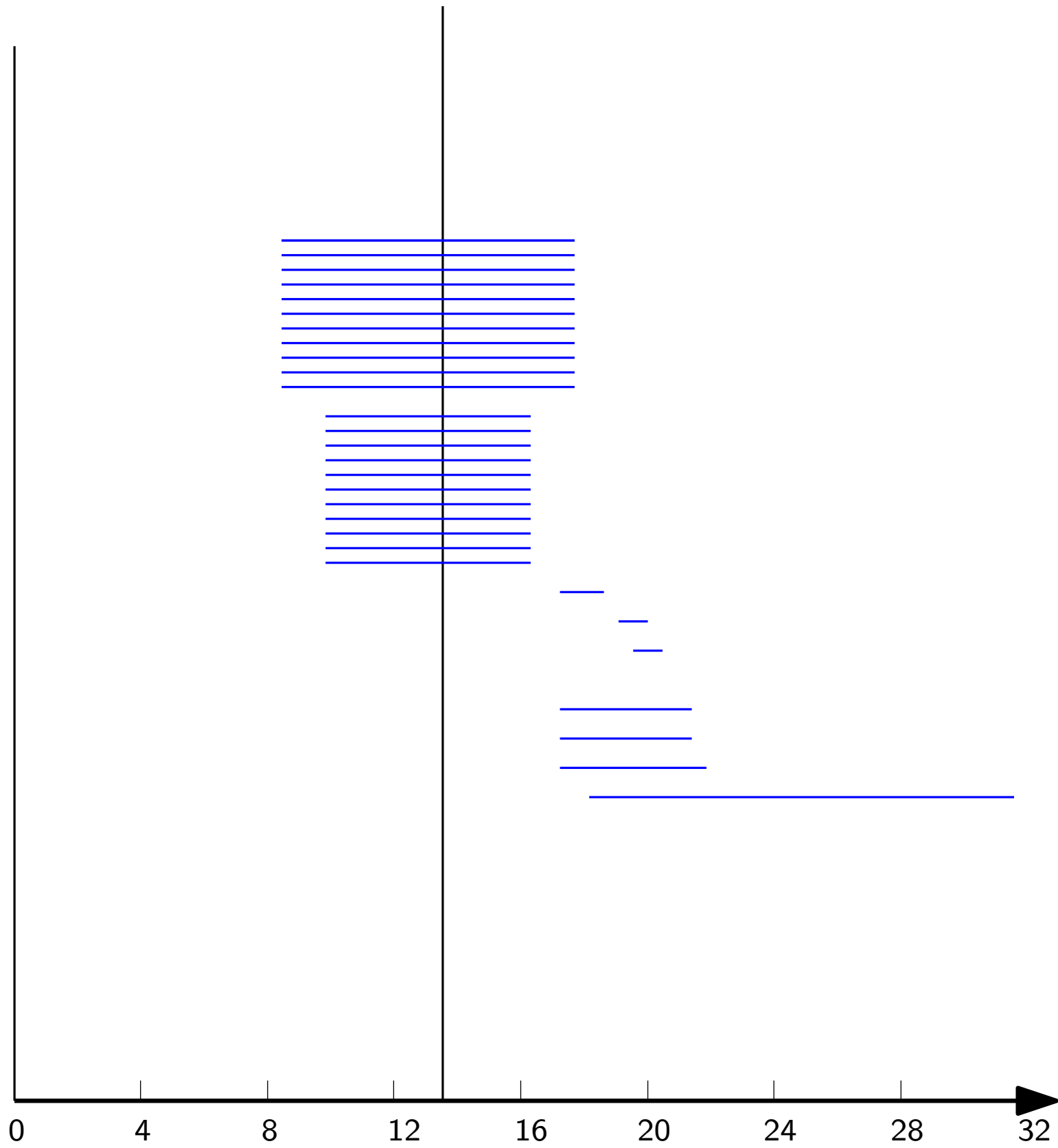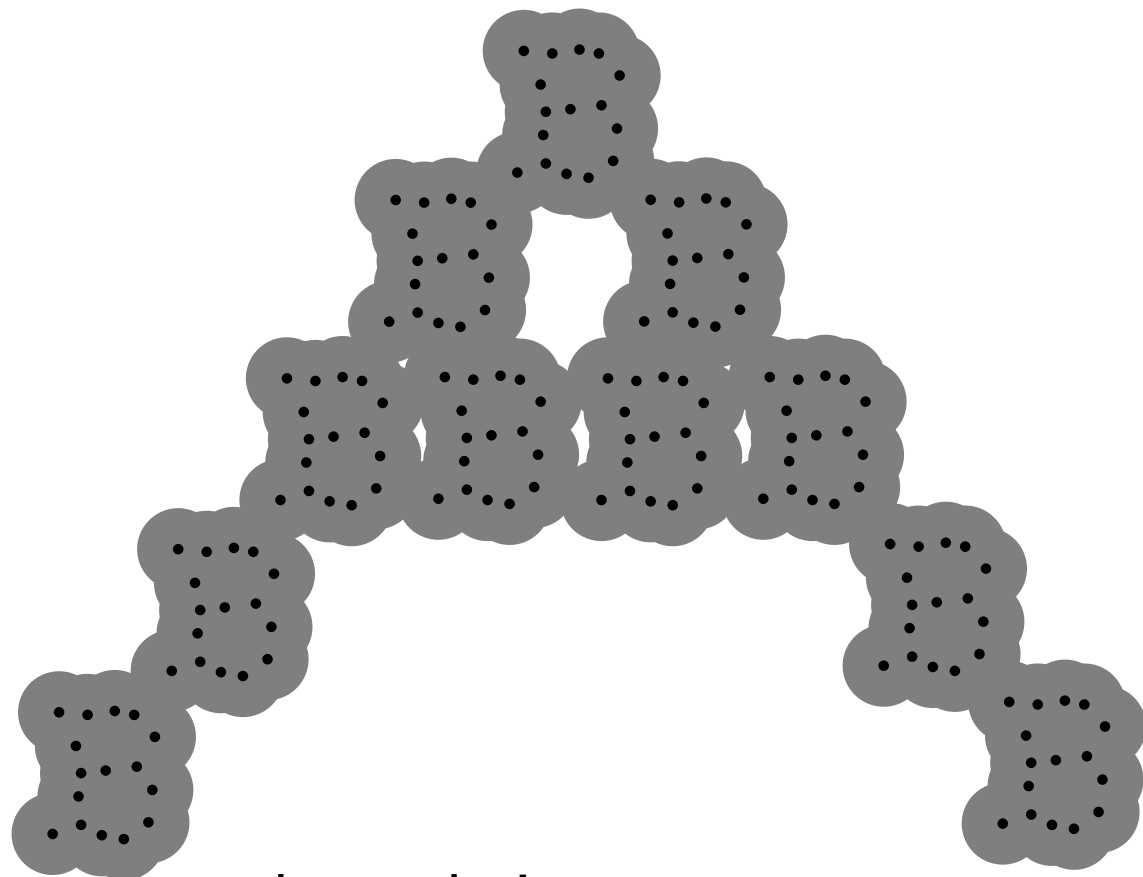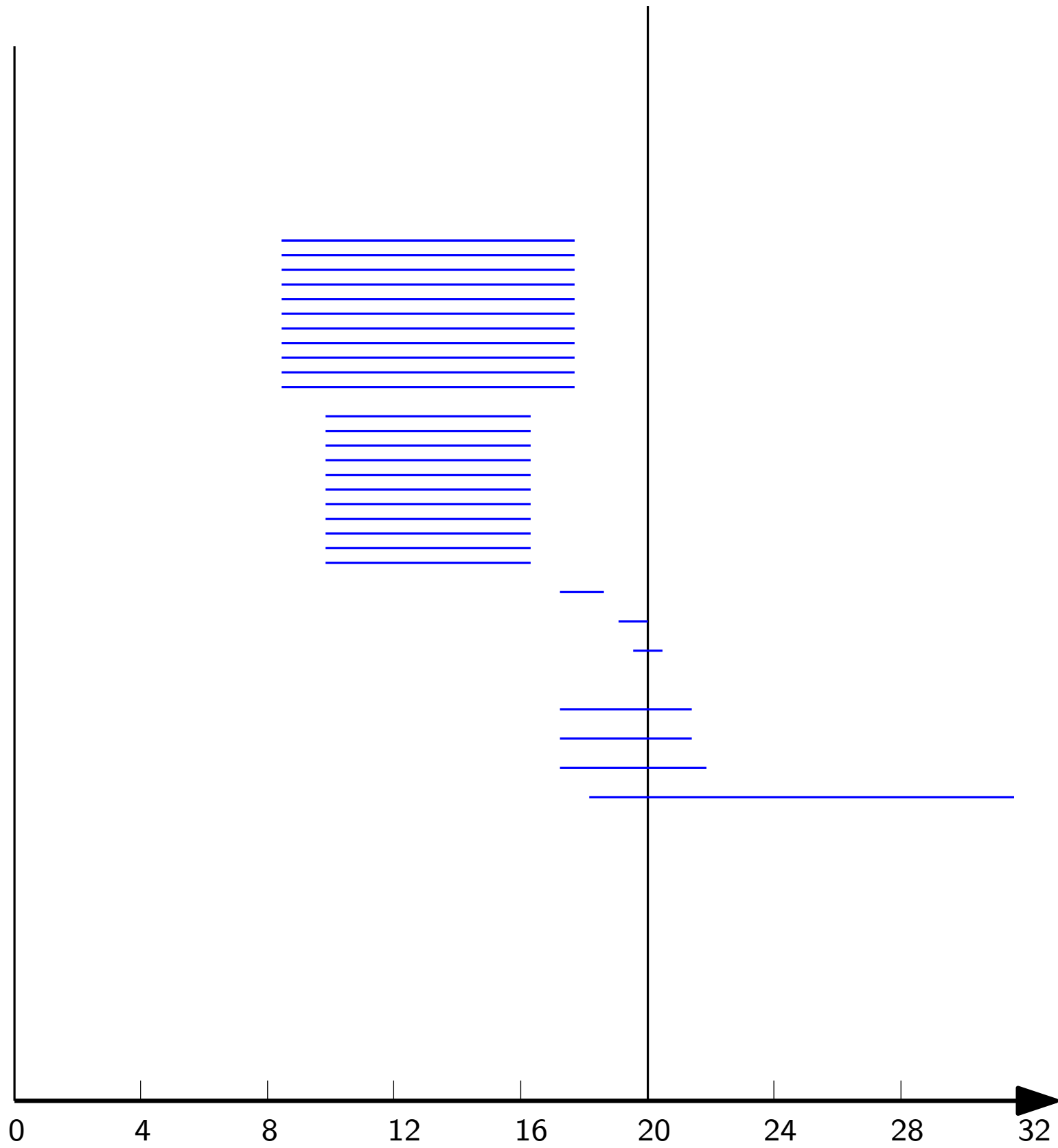- more stable
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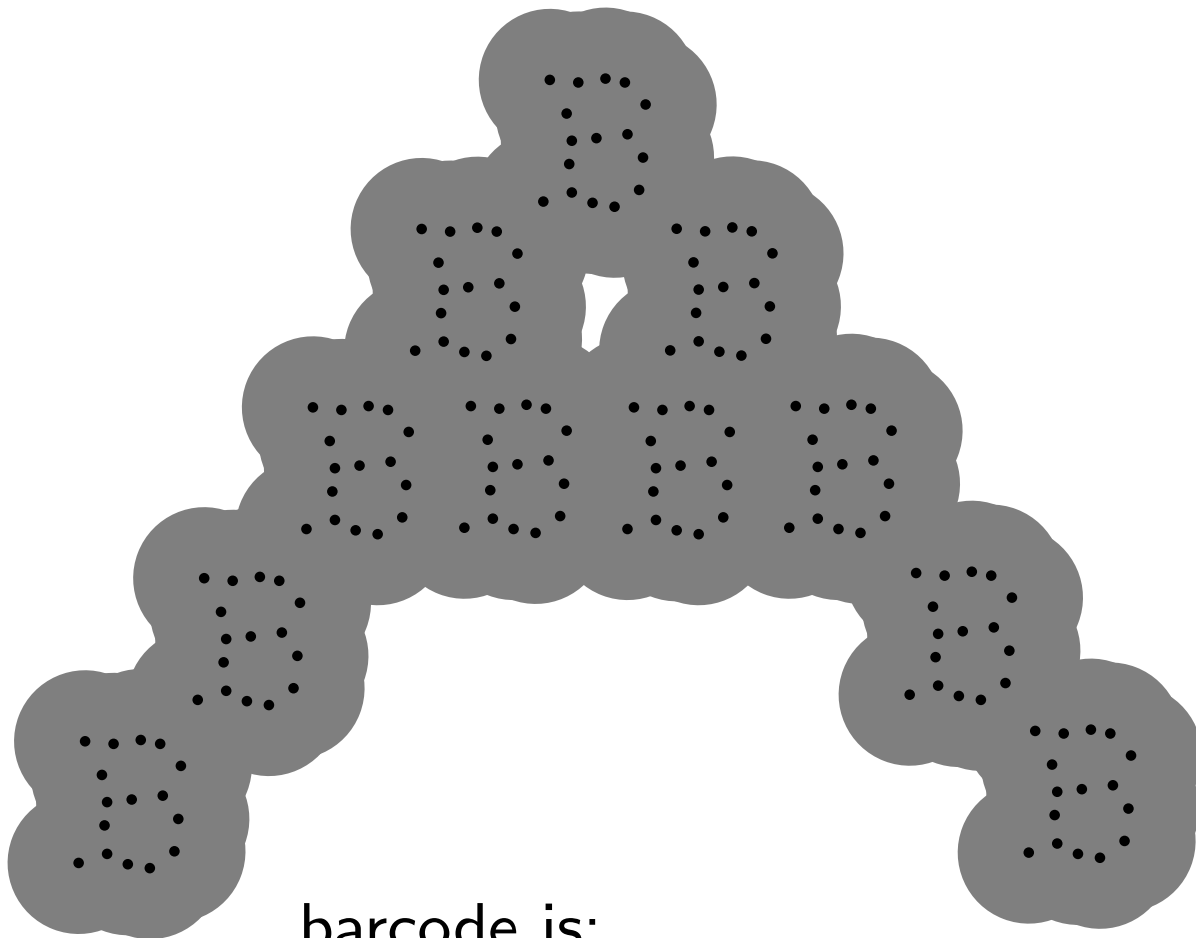$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$

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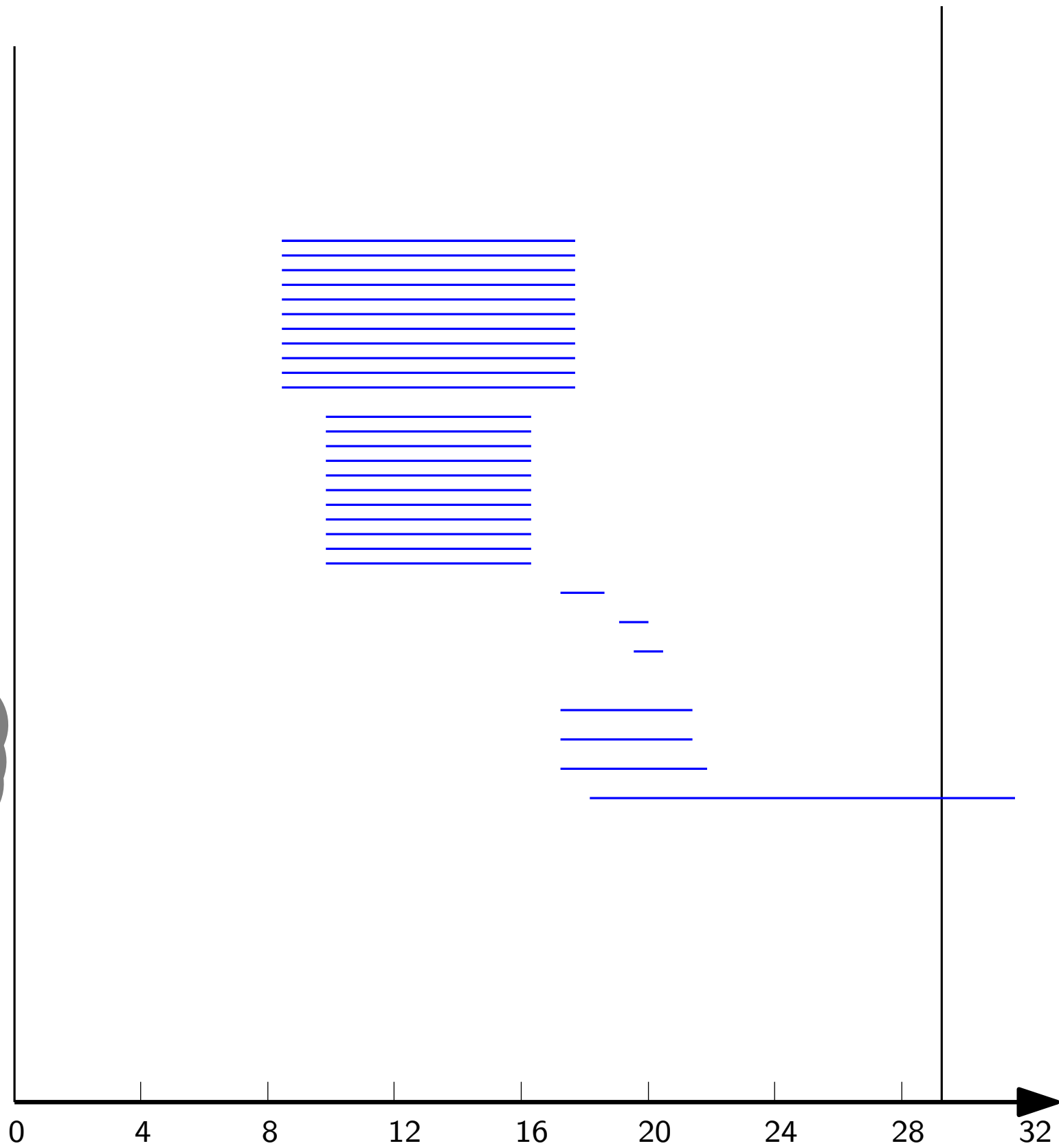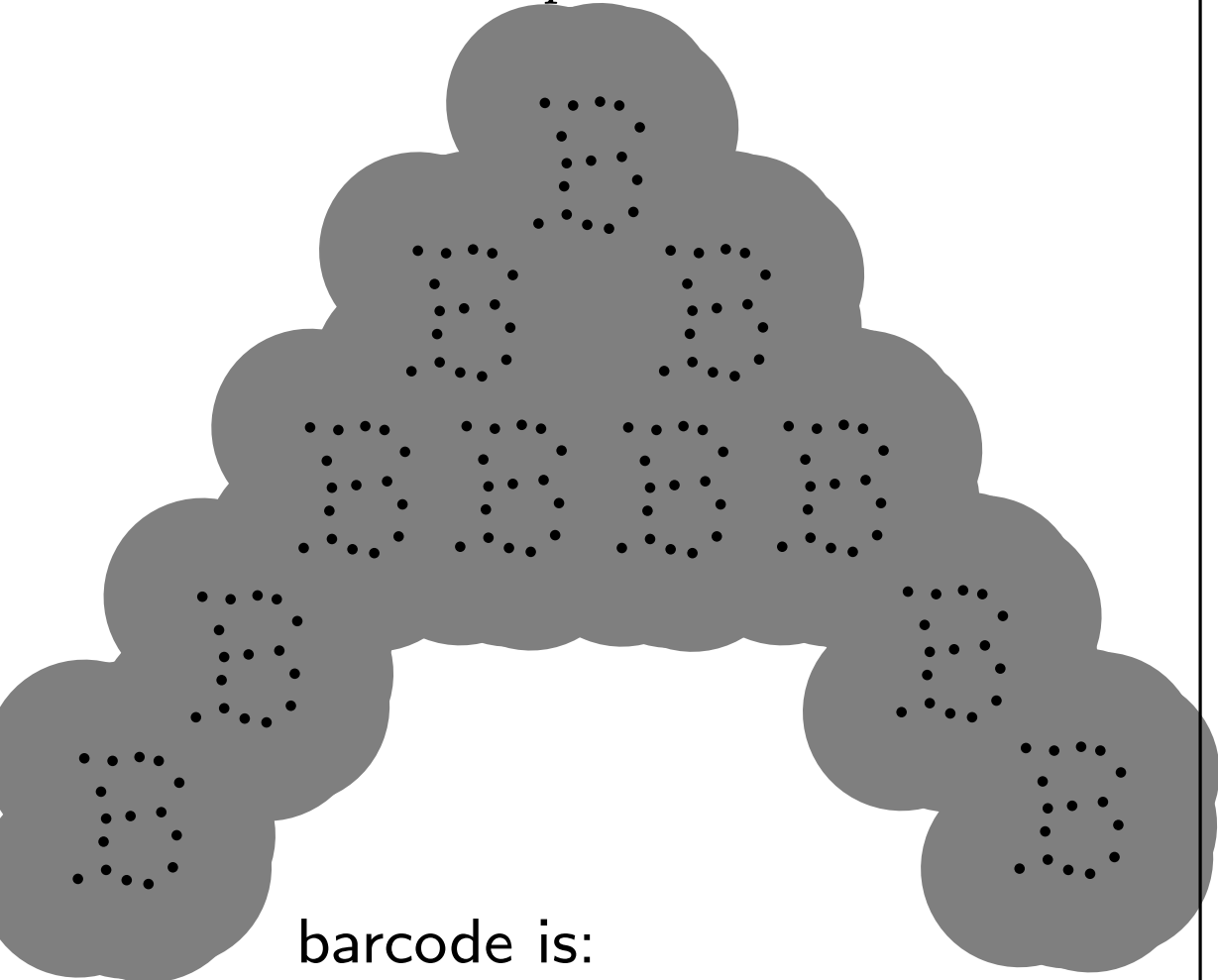- less (but still) informative
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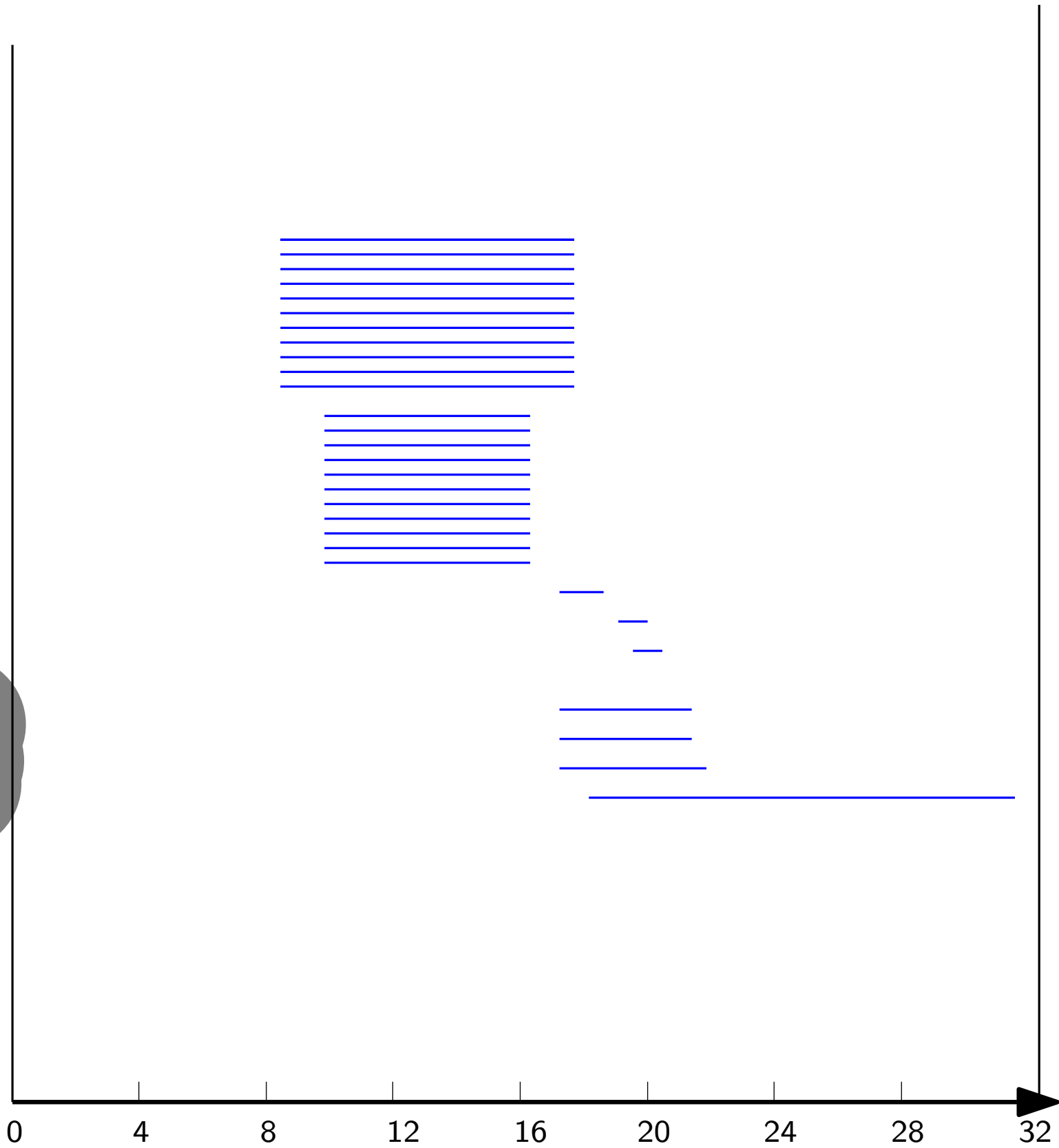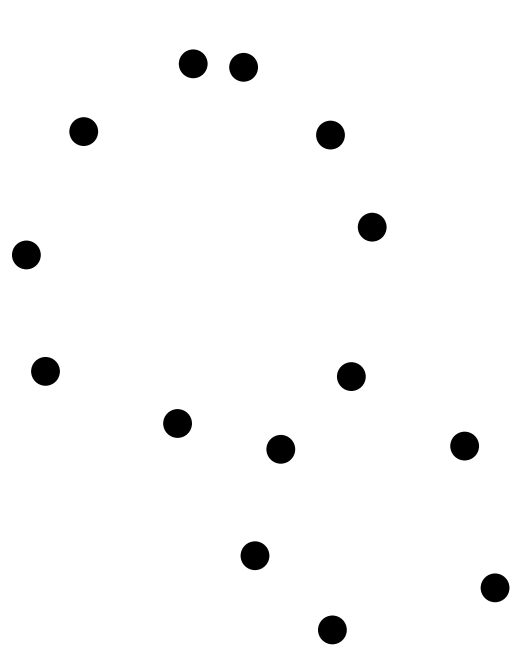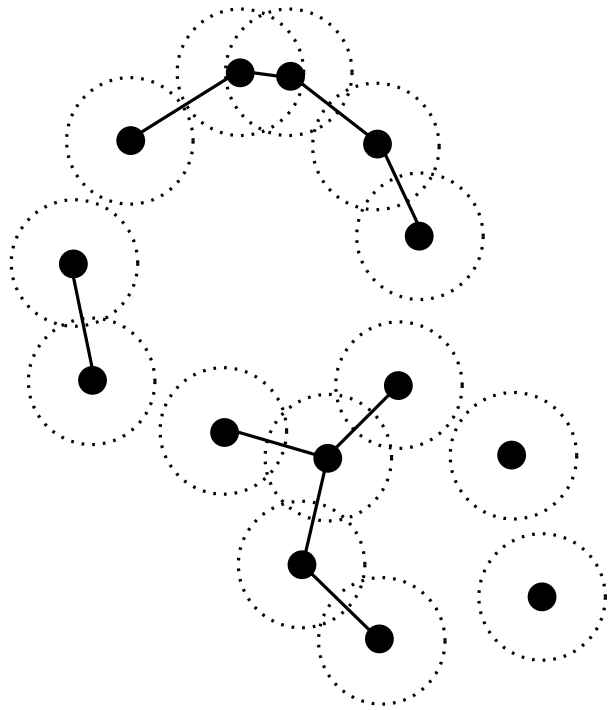- less (but still) informative
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Persistence for Point Cloud Data

$$\mathrm{d}_{\mathcal{P}} : \quad \mathbb{R}^2 \to \mathbb{R}$$
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Persistence for Point Cloud Data

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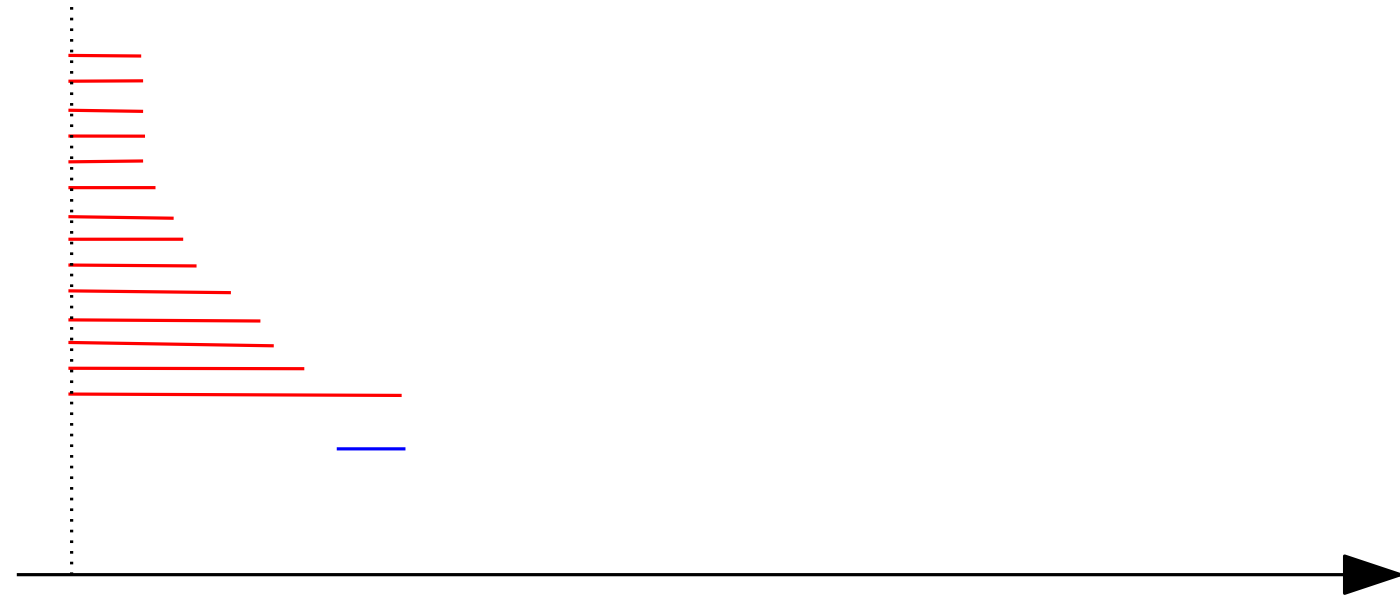$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$

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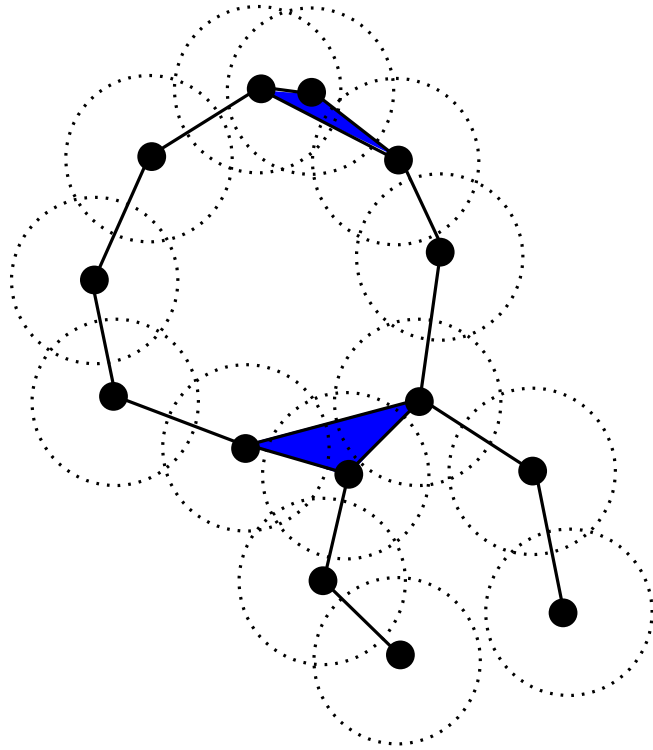
Persistence for Point Cloud Data

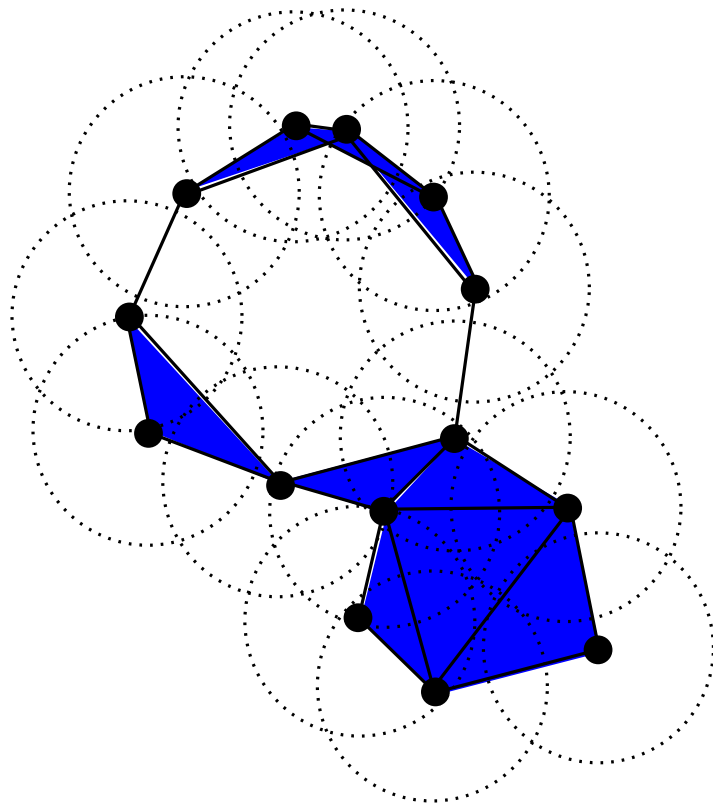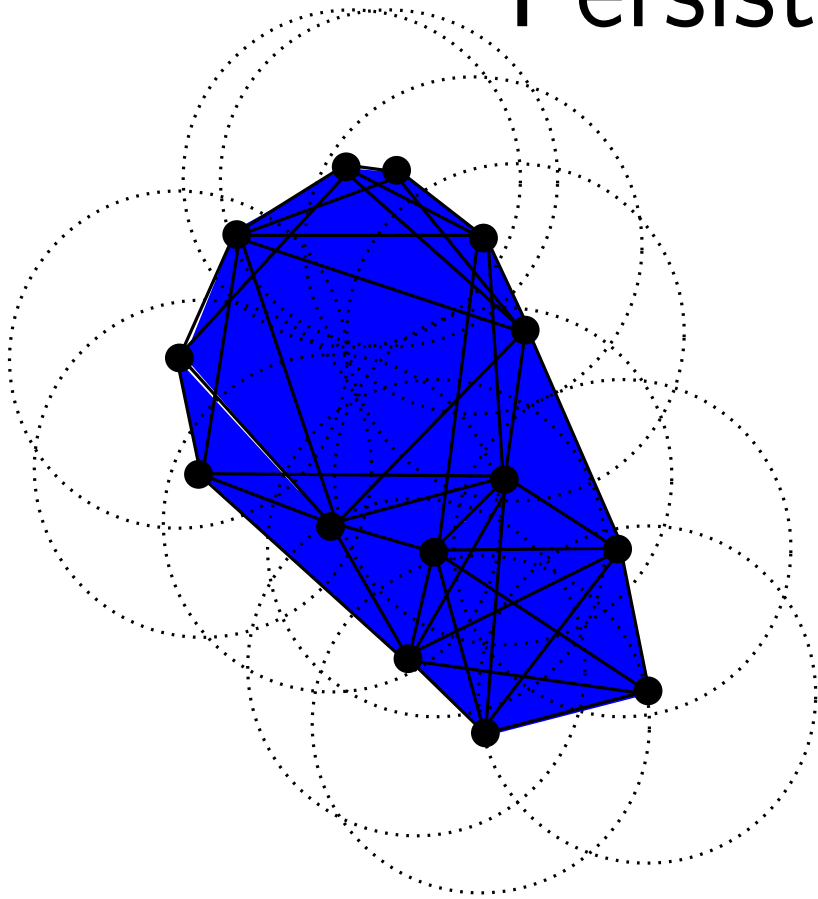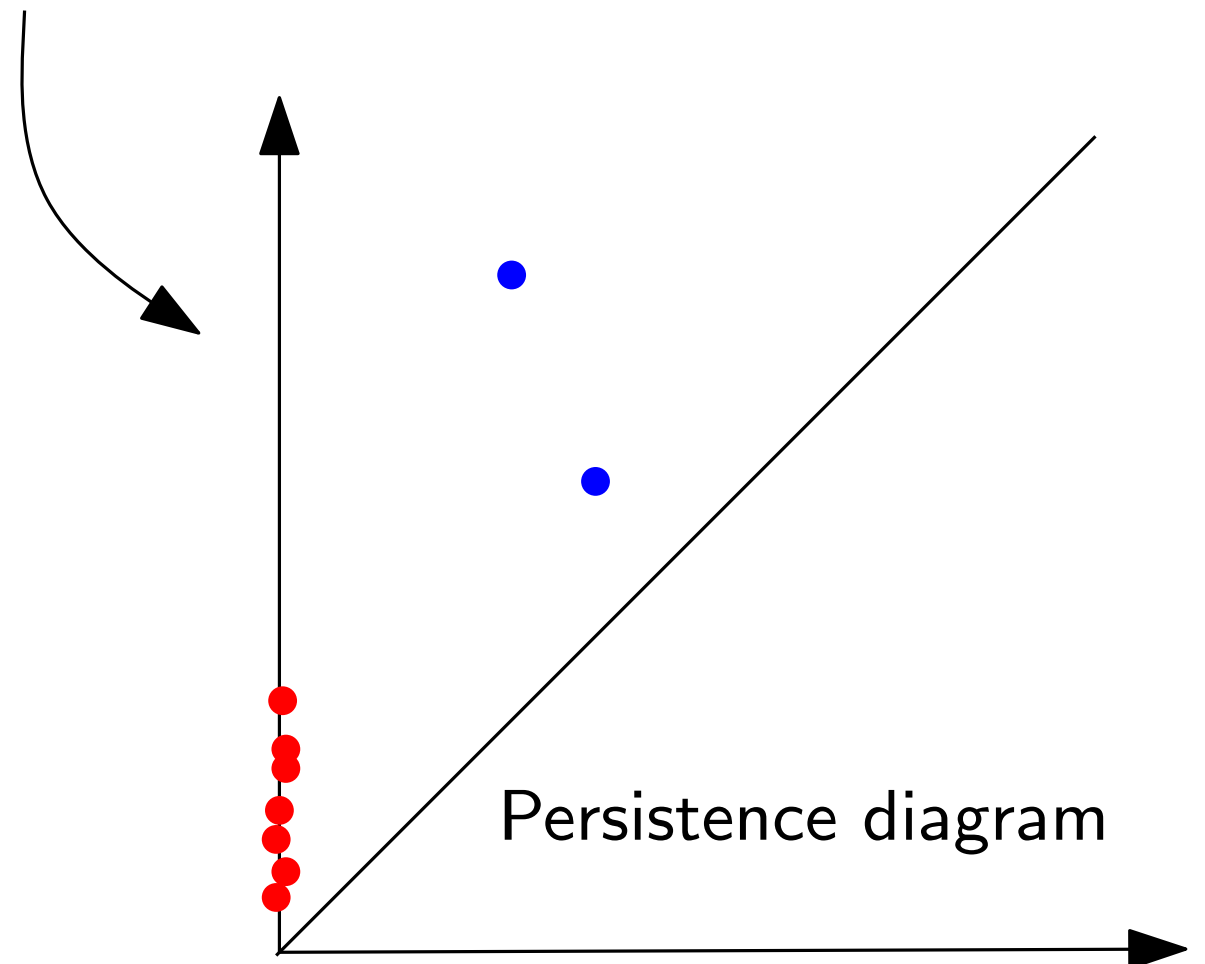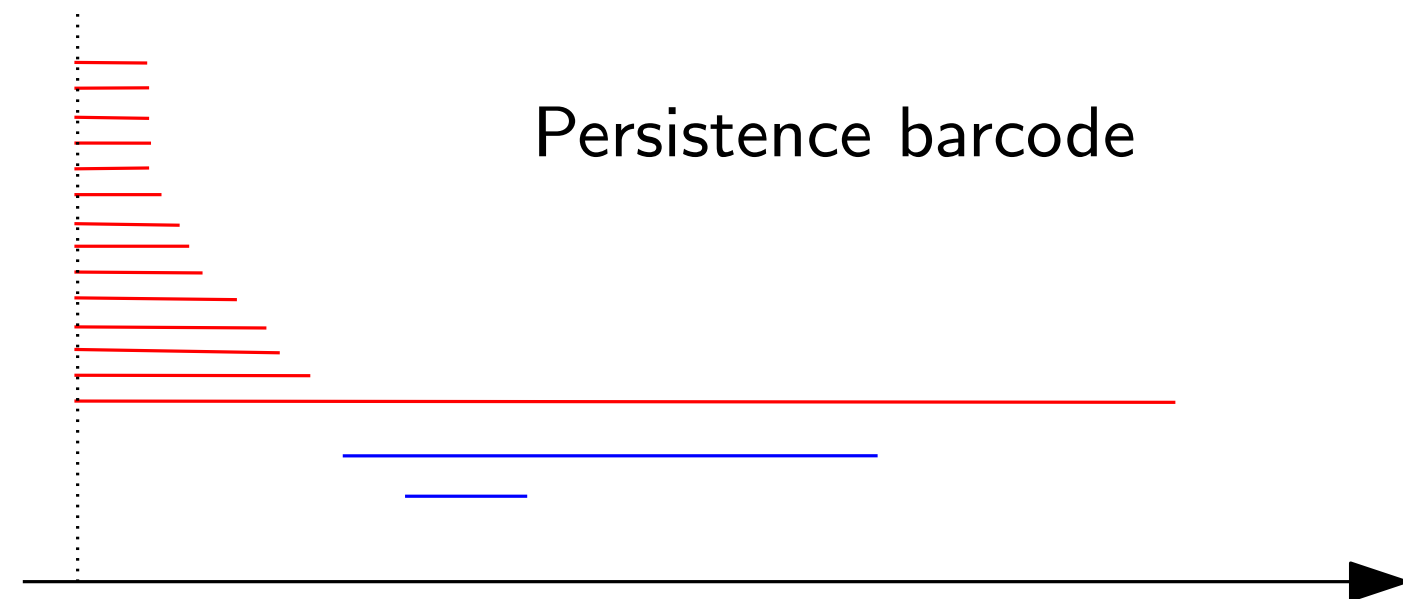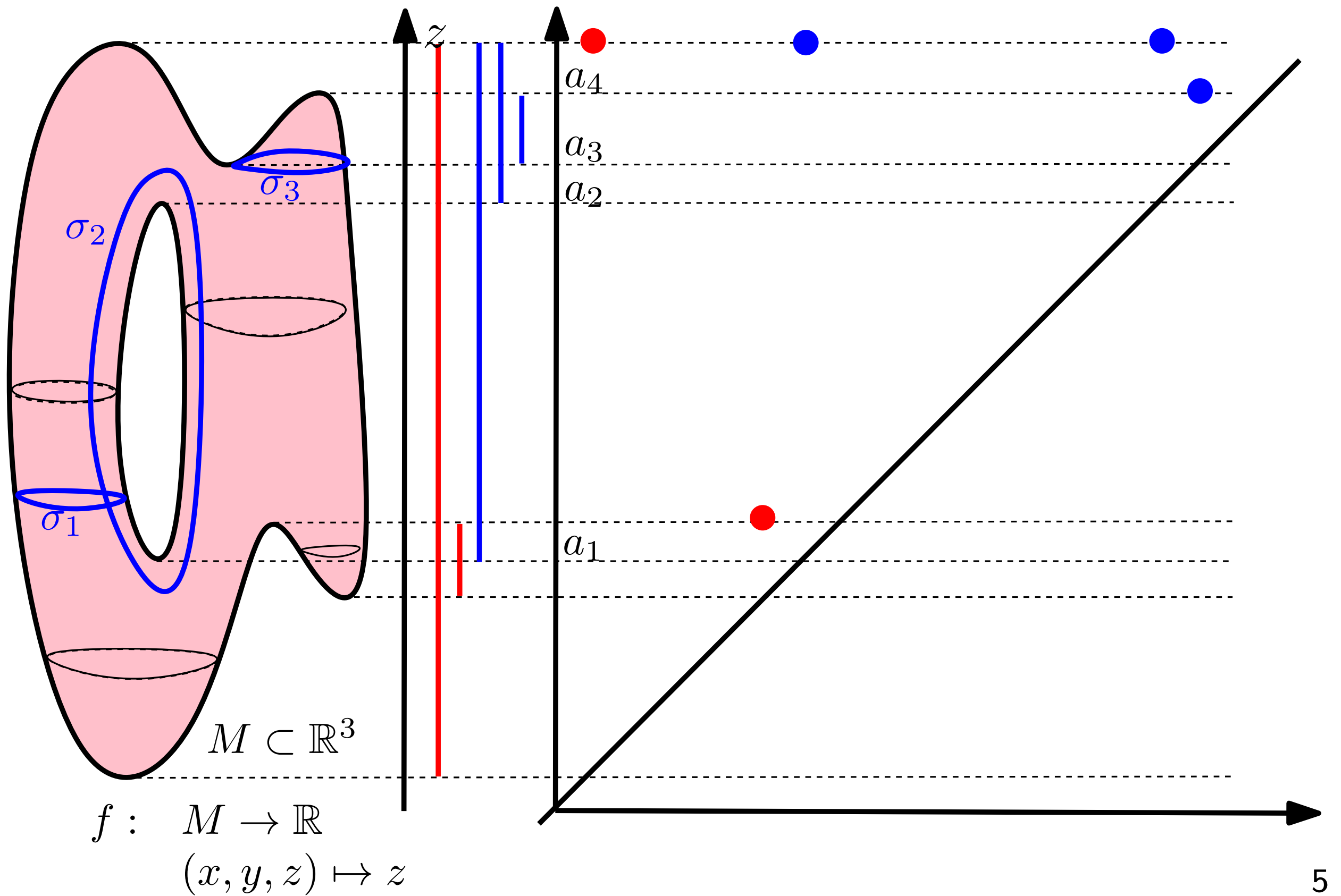$$\mathrm{d}_{\mathcal{P}}: \quad \mathbb{R}^2 \to \mathbb{R}$$
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4

Persistence for Point Cloud Data

Persistence barcode

$$\mathrm{d}_{\mathcal{P}} : \quad \mathbb{R}^2 \to \mathbb{R}$$
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Persistence diagram

Surface in Space

$M \subset \mathbb{R}^3$

z

σ_1

σ_2

σ_3

a_1

a_2

a_3

a_4

$f: \quad M \to \mathbb{R}$
$(x, y, z) \mapsto z$

Persistence Stability

Theorem: For any "nice" functions $f, g : X \to \mathbb{R}$,

$$\mathrm{d_b}(\mathrm{dgm}_k f, \mathrm{dgm}_k g) \le \|f - g\|_\infty$$